1. Populations of hydrogen in thermodynamic equilibrium

The aims of this problem are, first, to make sure you are figuring and plotting things on a computer, and second, to present you with an example of solving a quadratic. There is a numerically stable and a numerically unstable way to solve the quadratic that you should encounter in this problem, and you should of course use the stable solution.

(a) Compute and plot

Write a computer program (in IDL, Mathematica, Matlab, or the language of your choice) to compute the populations of the 1, 2, and 3 energy levels of neutral H (ignore higher levels), and of protons $p$, in thermodynamic equilibrium at a given temperature $T$ and a given total hydrogen number density

$$n_{\text{tot}} = n_H + n_p , \quad n_H = n_1 + n_2 + n_3 .$$

Assume overall charge neutrality

$$n_e = n_p .$$

Plot your results as a function of temperature $T$ for representative densities $n_{\text{tot}} = 10^{26}, 10^{28}, 10^{30},$ and $10^{32}$ m$^{-3}$. [Hint: In thermodynamic equilibrium, the number density $n_i$ of the $i$’th energy level of neutral hydrogen is given by the Boltzmann distribution

$$n_i = \lambda g_i e^{-\chi_i/kT} ,$$

where $\lambda$ is a normalization constant, $g_i = 4i^2$ is the degeneracy of level $i$ (the factor 4 comes from the 2 spin states of the electron, multiplied by the 2 spin states of the proton nucleus) and $\chi_i = -\chi/i^2$ is the energy of level $i$ relative to the energy of the just ionized ion, where $\chi$ is the ionization potential of hydrogen

$$\chi \approx 1 \text{ Rydberg} = 13.6 \text{ eV} = 157,800 \text{ K}$$

(the ionization potential $\chi = 13.5984$ eV is actually slightly less than 1 Rydberg $\equiv \frac{1}{2} \alpha^2 m_e c^2 = 13.60569193$ eV). The number densities $n_p$ of protons and $n_e$ of electrons in thermodynamic equilibrium with neutral hydrogen are given by the Saha equation

$$\frac{n_p n_e}{n_i} = \frac{g_p g_e}{g_i e^{-\chi_i/kT}} \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} ,$$

where $g_p = g_e = 2$ are the degeneracies (number of spin states) of protons and electrons.]

(b) Comment

Why is $n_{\text{tot}} \approx 10^{30}$ m$^{-3}$ an interesting choice? Is the calculation likely to be valid for densities much larger than this? [Hint: Roughly, what is the wavelength of an atomic electron, that is, an electron with energy of order $\chi$? What is the atomic (Bohr) radius?]