1. Forced, damped, simple harmonic oscillator

The problem is to solve the forced, damped, simple harmonic oscillator (overdot signifies differentiation \(d/dt\) with respect to time)

\[
\ddot{z} + 2a\dot{z} + k^2 z = f(t),
\]

subject to a sinusoidal forcing function that turns on at \(t = 0\),

\[
f(t) = \begin{cases} 
0 & (t \leq 0) \\
 c \sin(\omega t) & (t \geq 0) 
\end{cases}
\]

Here \(a\), \(k\), \(c\), and \(\omega\) are all constants. The forced, damped, simple harmonic oscillator equation is a prototype for equations that appear ubiquitously in the physical sciences, wherever there are small amplitude perturbations, such as waves of any kind, that have a natural frequency \(k\), are damped with damping constant \(a\), and are excited by some external forcing function.

(a) Homogeneous solutions

Find the homogeneous solutions to equation (1.1). Consider cases \(a < k\), \(a > k\), and \(a = k\). In the last case, \(a = k\), use the Wronskian to find a second solution. [Hint: Assume without loss of generality that \(k\) is positive. You may also assume that \(a\) is positive. Why is the damping constant \(a\) likely to be positive in physical situations? I found it convenient to introduce a quantity \(q \equiv \sqrt{k^2 - a^2}\).]

(b) Green’s function

Find the Green’s function \(G(t, t_0)\) of the damped simple harmonic oscillator, the solution to

\[
\ddot{G} + 2a\dot{G} + k^2 G = \delta_D(t - t_0)
\]

subject to the “retarded” boundary condition that \(G(t, t_0)\) vanishes for \(t < t_0\).

(c) Solution for oscillating forcing function

Use the Green’s function to solve equation (1.1) subject to the forcing function (1.2).

(d) Plot and comment

Plot and comment on a representative sample of solutions with various \(a\), \(k\), \(c\), and \(\omega\). Show that, after a long time, the solution oscillates with amplitude

\[
\frac{c}{\sqrt{(k^2 - \omega^2)^2 + (2a\omega)^2}}
\]

What happens “near resonance”, whatever that means? [Hint: In thinking about solutions with various \(a\), \(k\), \(c\), and \(\omega\), you can scale to \(k = 1\) and \(c = 1\) without loss of generality. Why?]