ASTR 3740 Relativity & Cosmology Spring 2004. Problem Set 5.
Due Wed 17 Mar

1. Geodesics in the Reissner-Nordström geometry

The Reissner-Nordström metric describes the geometry of empty space in and around a spherically symmetric black hole of mass M and charge Q. In units $c = G = 1$, the metric is

$$ds^2 = B\,dt^2 - \frac{dr^2}{B} - r^2 do^2 \tag{1.1}$$

where $do^2 \equiv d\theta^2 + \sin^2\theta\,d\phi^2$ is the metric on the surface of a unit 3-sphere, and

$$B = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \ . \tag{1.2}$$

Similarly to Problem Set 4, the equations of motion of a (neutral) particle freely-falling in the Reissner-Nordström geometry are

$$\begin{aligned} B\frac{dt}{ds} &= E \\ r^2\frac{d\phi}{ds} &= L \\ \left(\frac{dr}{ds}\right)^2 + V_{\rm eff}^2 &= E^2 \end{aligned} \tag{1.3}$$

where s is the proper time of the particle, and E and L are constants, the particle's energy and angular momentum per unit mass. The quantity $V_{\rm eff}$ is the effective potential given by

$$V_{\rm eff}^2 = B\left(1 + \frac{L^2}{r^2}\right) \ . \tag{1.4}$$

(a) Horizons

Horizons in the RN geometry occur where a worldline that is at rest in the geometry, $dr = d\theta = d\phi = 0$, is also a null geodesic, $ds = 0$. What is the condition on the metric coefficient B for a horizon to occur?

For the RN geometry, what are the radii of the horizons in terms of the mass M and charge Q? Evaluate these radii, in units of the BH mass M, for the case where $Q/M = 0.8$.

What condition on the charge to mass ratio Q/M of the BH is necessary for horizons to exist? FYI, the critical case is called an extremal black hole, which proves to be a case of special interest — for example, the innermost circular orbit of a charged particle with the same charge to mass as the BH is at the horizon, for an extremal BH.

(b) Radial free-faller

A person who falls radially from zero velocity at infinity has unit energy per unit mass, $E = 1$, and zero angular momentum per unit mass, $L = 0$. Why? [Hint: Impose the condition of zero velocity on the equations of motion (1.3) in the limit $r \to \infty$.]

Denote the proper time experienced by such a radial free-faller by t_{ff}, so that $t_{\text{ff}} = s$ along the worldline of the free-faller. The free-faller changes their radial position r in a proper time t_{ff} at free-fall velocity

$$\text{v} \equiv -\frac{dr}{dt_{\text{ff}}} \ . \tag{1.5}$$

What is this velocity v in terms of the metric coefficient B?

What is the value of the free-fall velocity at a horizon? There are two possible signs to this value, one corresponding to a black hole, the other to a white hole. Which is which?

In the RN geometry, at what radius r_0, the turnaround radius, does the free-fall velocity v go to zero, besides $r \to \infty$?

Plot the free-fall velocity v as a function of radius r for the case $Q/M = 0.8$. Don't forget the two possible signs of the square root.

Using your plot of the velocity v as a guide, describe in words the trip that the radial free-faller has through the BH.

No credit: Integrate to obtain an explicit expression for the free-fall time $t_f f$ as a function of radius r.

(c) River model

Show that the coordinate transformation

$$dt = dt_{\text{ff}} - \frac{\text{v}}{1 - \text{v}^2}\, dr \tag{1.6}$$

transforms the metric (1.1) into the river metric

$$ds^2 = dt_{\text{ff}}^2 - (dr + \text{v}\, dt_{\text{ff}})^2 - r^2 do^2 \ . \tag{1.7}$$

[Hint: It is easiest to derive this by expressing the metric coefficient B in terms of v.]

(d) Zero energy geodesic

Return to the equations of motion (1.3) and consider the case of a geodesic with zero energy and angular momentum, What is the radial velocity dr/ds on this orbit?

What are the minimum and maximum radii of the geodesic, where the velocity goes to zero?

No credit: Integrate to find an explicit expression for the proper time s as a function of radius r on this orbit.

(e) Penrose diagram

Sketch a Penrose diagram of the RN geometry, and on it sketch the trajectories of the two cases you have considered, radial free-fallers with $E = 1$ and $E = 0$ respectively.