

**ASTR 3740 Relativity & Cosmology Spring 2007. Problem Set 5.**  
**Due Wed 21 Mar**

**1. Geodesics in the Reissner-Nordström geometry**

The Reissner-Nordström metric describes the geometry of empty space in and around a spherically symmetric black hole of mass  $M$  and charge  $Q$ . In units  $c = G = 1$ , the metric is

$$ds^2 = B dt^2 - \frac{dr^2}{B} - r^2 d\sigma^2 \quad (1.1)$$

where  $d\sigma^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$  is the metric on the surface of a unit 3-sphere, and

$$B = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} . \quad (1.2)$$

Similarly to Problem Set 4, the equations of motion of a (neutral) particle freely-falling in the Reissner-Nordström geometry are

$$\begin{aligned} B \frac{dt}{ds} &= E \\ r^2 \frac{d\phi}{ds} &= L \\ \left( \frac{dr}{ds} \right)^2 + V_{\text{eff}}^2 &= E^2 \end{aligned} \quad (1.3)$$

where  $s$  is the proper time of the particle, and  $E$  and  $L$  are constants, the particle's energy and angular momentum per unit mass. The quantity  $V_{\text{eff}}$  is the effective potential given by

$$V_{\text{eff}}^2 = B \left( 1 + \frac{L^2}{r^2} \right) . \quad (1.4)$$

**(a) Horizons**

Horizons in the RN geometry occur where a worldline that is at rest in the geometry,  $dr = d\theta = d\phi = 0$ , is also a null geodesic,  $ds = 0$ . What is the condition on the metric coefficient  $B$  for a horizon to occur?

For the RN geometry, what are the radii of the horizons in terms of the mass  $M$  and charge  $Q$ ? Evaluate these radii, in units of the BH mass  $M$ , for the case where  $Q/M = 0.8$ .

What condition on the charge to mass ratio  $Q/M$  of the BH is necessary for horizons to exist? FYI, the critical case is called an extremal black hole, which proves to be a case of special interest — for example, the innermost circular orbit of a charged particle with the same charge to mass as the BH is at the horizon, for an extremal BH.

### (b) Radial free-faller

A person who falls radially from zero velocity at infinity has unit energy per unit mass,  $E = 1$ , and zero angular momentum per unit mass,  $L = 0$ . Why? [Hint: Impose the condition of zero velocity on the equations of motion (1.3) in the limit  $r \rightarrow \infty$ .]

Denote the proper time experienced by such a radial free-faller by  $t_{\text{ff}}$ , so that  $t_{\text{ff}} = s$  along the worldline of the free-faller. The free-faller changes their radial position  $r$  in a proper time  $t_{\text{ff}}$  at free-fall velocity

$$v \equiv -\frac{dr}{dt_{\text{ff}}} . \quad (1.5)$$

What is this velocity  $v$  in terms of the metric coefficient  $B$ ?

What is the value of the free-fall velocity at a horizon? There are two possible signs to this value, one corresponding to a black hole, the other to a white hole. Which is which?

In the RN geometry, at what radius  $r_0$ , the turnaround radius, does the free-fall velocity  $v$  go to zero, besides  $r \rightarrow \infty$ ?

Plot the free-fall velocity  $v$  as a function of radius  $r$  for the case  $Q/M = 0.8$ . Don't forget the two possible signs of the square root.

Using your plot of the velocity  $v$  as a guide, describe in words the trip that the radial free-faller has through the BH.

No credit: Integrate to obtain an explicit expression for the free-fall time  $t_{\text{ff}}$  as a function of radius  $r$ .

### (c) River model

Show that the coordinate transformation

$$dt = dt_{\text{ff}} - \frac{v}{1 - v^2} dr \quad (1.6)$$

transforms the metric (1.1) into the river metric

$$ds^2 = dt_{\text{ff}}^2 - (dr + v dt_{\text{ff}})^2 - r^2 d\sigma^2 . \quad (1.7)$$

[Hint: It is easiest to derive this by expressing the metric coefficient  $B$  in terms of  $v$ .]

### (d) Zero energy geodesic

Return to the equations of motion (1.3) and consider the case of a geodesic with zero energy and angular momentum, What is the radial velocity  $dr/ds$  on this orbit?

What are the minimum and maximum radii of the geodesic, where the velocity goes to zero?

Plot the radial velocity  $dr/ds$  on a diagram.

No credit: Integrate to find an explicit expression for the proper time  $s$  as a function of radius  $r$  on this orbit.

### (e) Penrose diagram

Sketch a Penrose diagram of the RN geometry, and on it sketch the trajectories of the two cases you have considered, radial free-fallers with  $E = 1$  and  $E = 0$  respectively.