ASTR 5540 Math Meth Fall 2008. Problem Set 1. Due Wed Sep 3

1. Populations of hydrogen in thermodynamic equilibrium

The aims of this problem are, first, to make sure you are figuring and plotting things on a computer, and second, to present you with an example of solving a quadratic. There is a numerically stable and a numerically unstable way to solve the quadratic that you should encounter in this problem, and you should of course use the stable solution.

(a) Compute and plot

Write a computer program (in IDL, Mathematica, Matlab, or the language of your choice) to compute the populations of the 1, 2, and 3 energy levels of neutral H (ignore higher levels), and of protons p, in thermodynamic equilibrium at a given temperature T and a given total hydrogen number density

$$n_{\rm tot} = n_{\rm H} + n_p \ , \quad n_{\rm H} = n_1 + n_2 + n_3 \ . \tag{1.1}$$

Assume overall charge neutrality

$$n_e = n_p \ . \tag{1.2}$$

Plot your results as a function of temperature T for representative densities $n_{\rm tot} = 10^{26}$, 10^{28}, 10^{30}, and $10^{32}\,{\rm m}^{-3}$. [Hint: In thermodynamic equilibrium, the number density n_i of the i'th energy level of neutral hydrogen is given by the Boltzmann distribution

$$n_i = \lambda \, g_i \, e^{-\chi_i/kT} \ , \tag{1.3}$$

where λ is a normalization constant, $g_i = 4\,i^2$ is the degeneracy of level i (the factor 4 comes from the 2 spin states of the electron, multiplied by the 2 spin states of the proton nucleus) and $\chi_i = -\chi/i^2$ is the energy of level i relative to the energy of the just ionized ion, where χ is the ionization potential of hydrogen

$$\chi \approx 1\,\text{Rydberg} = 13.6\,\text{eV} = 157{,}800\,\text{K} \tag{1.4}$$

(the ionization potential $\chi = 13.5984\,$eV is actually slightly less than 1 Rydberg $\equiv \frac{1}{2}\alpha^2 m_e c^2 = 13.60569193\,$eV). The number densities n_p of protons and n_e of electrons in thermodynamic equilibrium with neutral hydrogen are given by the Saha equation

$$\frac{n_p n_e}{n_i} = \frac{g_p g_e}{g_i e^{-\chi_i/kT}} \left(\frac{2\pi m_e k T}{h^2} \right)^{3/2} \ , \tag{1.5}$$

where $g_p = g_e = 2$ are the degeneracies (number of spin states) of protons and electrons.]

(b) Comment

Why is $n_{\rm tot} \approx 10^{30}\,{\rm m}^{-3}$ an interesting choice? Is the calculation likely to be valid for densities much larger than this? [Hint: Roughly, what is the wavelength of an atomic electron, that is, an electron with energy of order χ? What is the atomic (Bohr) radius?]