ASTR 5540 Math Meth Fall 2008. Problem Set 10. Due Mon Nov 10

1. Riemann Problem

The Riemann problem provides the physical basis for the shock-capturing "Riemann solvers" that are at the heart of any modern hydrodynamic code. The Riemann problem concerns a fluid of adiabatic index γ, evolving in one dimension from initial conditions of uniform velocity v_l, sound speed c_l, and pressure p_l on the left, $x < 0$, and uniform velocity v_r, sound speed c_r, and pressure p_r on the right, $x > 0$. The evolution in time t and space x of the density ρ, velocity v, and pressure p of the fluid are governed by the equations of conservation of mass and momentum

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} &= 0 \ , \\ \frac{\partial \rho v}{\partial t} + \frac{\partial (\rho v^2 + p)}{\partial x} &= 0 \ . \end{aligned} \tag{1.1}$$

The energy conservation equation is equivalent to

$$p = K(s)\, \rho^\gamma \ , \tag{1.2}$$

where s is the entropy per unit mass. The sound speed c is given by

$$c^2 \equiv \left.\frac{dp}{d\rho}\right|_s = \frac{\gamma p}{\rho} \ . \tag{1.3}$$

(a) Isentropic

In isentropic regions, where the entropy s per unit mass is constant, the parameter $K(s)$ in equation (1.2) is constant. Show that in isentropic regions the fluid equations (1.1) can be written

$$\begin{aligned} \frac{\partial c}{\partial t} + v \frac{\partial c}{\partial x} + \frac{\gamma - 1}{2} c \frac{\partial v}{\partial x} &= 0 \ , \\ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial x} &= 0 \ . \end{aligned} \tag{1.4}$$

(b) Eigensystem

Rewrite equations (1.4) in the form

$$\frac{\partial \boldsymbol{y}}{\partial t} + A \frac{\partial \boldsymbol{y}}{\partial x} = 0 \tag{1.5}$$

where $\boldsymbol{y}$ is the vector

$$\boldsymbol{y} \equiv \begin{pmatrix} c \\ v \end{pmatrix} \tag{1.6}$$

and A is a matrix. What are the eigenvalues and eigenvectors of the matrix A? Conclude that the isentropic equations (1.4) can be written

$$\left[\frac{\partial}{\partial t} + (v \pm c)\frac{\partial}{\partial x}\right]\left(v \pm \frac{2c}{\gamma - 1}\right) = 0 \ . \tag{1.7}$$

(c) Self-similar

The initial conditions of the Riemann problem are self-similar, all quantities with dimension of velocity being constant. It follows that the Riemann problem admits a self-similar solution in which all quantities with dimension velocity are functions only of the velocity variable $\xi \equiv x/t$. Argue that, for such a self-similar solution, the isentropic equations (1.7) can be written

$$(-\xi + v \pm c)\frac{d}{d\xi}\left(v \pm \frac{2c}{\gamma - 1}\right) = 0 \ . \tag{1.8}$$

[Hint: In such a self-similar solution, v and c are functions only of $\xi \equiv x/t$.] Conclude from equations (1.8) that, in isentropic regions,

$$\text{either} \quad \xi = v + c \quad \text{or} \quad v + \frac{2c}{\gamma - 1} = \text{constant} \ , \tag{1.9}$$

and in addition

$$\text{either} \quad \xi = v - c \quad \text{or} \quad v - \frac{2c}{\gamma - 1} = \text{constant} \ . \tag{1.10}$$

Hence conclude that, in isentropic regions where the sound speed c is non-vanishing, one of the following three conditions holds:

$$\begin{array}{lll} v = \text{constant} & \text{and} & c = \text{constant} \ ; \\ \xi = v + c & \text{and} & v - \dfrac{2c}{\gamma - 1} = \text{constant} \ ; \\ \xi = v - c & \text{and} & v + \dfrac{2c}{\gamma - 1} = \text{constant} \ . \end{array} \tag{1.11}$$

(d) Rarefaction wave to the left

The left and right sides of the fluid abut at a contact interface moving with constant velocity v_c. One possibility is that the region to the left of the contact interface, $\xi \leq v_c$, is a rarefaction wave. In this case there is a region where the last of the three conditions (1.11) holds, encompassed by regions where the velocity v and sound speed c are constant:

$$\begin{array}{llll} v = v_l & \text{and} \quad c = c_l & & \text{for} \quad \xi \leq v_l - c_l \ , \\ \xi = v - c & \text{and} \quad v + \dfrac{2c}{\gamma - 1} = \text{constant} & & \text{for} \quad v_l - c_l \leq \xi \leq v_c - c_{cl} \ , \\ v = v_c & \text{and} \quad c = c_{cl} & & \text{for} \quad v_c - c_{cl} \leq \xi \leq v_c \ . \end{array} \tag{1.12}$$

Obtain expressions for v and c as a function of ξ in this region. Obtain an expression for the constant c_{cl} in terms of the constants v_c, v_l, and c_l. Argue that the condition

$$v_l - c_l \leq v_c - c_{cl} \tag{1.13}$$

is true if and only if

$$v_l \leq v_c \ . \tag{1.14}$$

Argue that the pressure p in the rarefaction wave is given by

$$\frac{p}{p_l} = \left(\frac{c}{c_l}\right)^{\frac{2\gamma}{\gamma-1}} \ , \tag{1.15}$$

and hence that the pressure p_c at the contact interface satisfies

$$\frac{p_c}{p_l} = \left(\frac{c_{cl}}{c_l}\right)^{\frac{2\gamma}{\gamma-1}} \ . \tag{1.16}$$

(e) Shock wave to the left

If condition (1.14) is not satisfied, then the region to the left of the contact interface, $\xi \leq v_c$, is a shock wave, not a rarefaction wave. In this case there is no region where either of the last two of the three conditions (1.11) holds. The velocity v and sound speed c are simply constant either side of the shock front:

$$\begin{aligned} v = v_l \quad &\text{and} \quad c = c_l \quad \text{for} \quad \xi \leq \xi_l \ , \\ v = v_c \quad &\text{and} \quad c = c_{cl} \quad \text{for} \quad \xi_l \leq \xi \leq v_c \ , \end{aligned} \tag{1.17}$$

where ξ_l is the velocity at which the shock front is moving. The Mach number M_l of the shock is, by its definition,

$$M_l = \frac{v_l - \xi_l}{c_l} \ , \tag{1.18}$$

which yields an expression for the velocity ξ_l of the shock in terms of v_l, c_l, and M_l. From the shock jump condition

$$\frac{v_l - v_c}{c_l} = \frac{2(M_l^2 - 1)}{(\gamma+1)M_l} \tag{1.19}$$

derive an expression for the Mach number M_l in terms of $(v_l - v_c)/c_l$. What is the sign of the square root in your expression, given that the Mach number must be ≥ 1? From the shock jump conditions for the density ρ and pressure p

$$\frac{\rho_l}{\rho_{cl}} = 1 - \frac{2(M_l^2 - 1)}{(\gamma+1)M_l^2} \ , \quad \frac{p_c}{p_l} = 1 + \frac{2\gamma(M_l^2 - 1)}{\gamma+1} \ , \tag{1.20}$$

derive an expression for the ratio c_{cl}/c_l of the post-shock to pre-shock sound speeds in terms of the Mach number M_l. [Hint: Recall the defintion (1.3) of the sound speed c.]

(f) Rarefaction or shock wave to the right

Argue that the region to the right of the contact interface, $\xi \geq v_c$, satisfies equations obtained from the equations for the left region by the transformations

$$l \to r \ , \quad \xi \to -\xi \ , \quad v \to -v \ . \tag{1.21}$$

Note that the velocity v_c and pressure p_c at the contact interface must be the same on both left and right sides, but the sound speeds c_{cl} and c_{cr} at the contact interface may differ between left and right sides.

(g) Plot

Plot the solution for the velocity v and sound speed c in the case that

$$\gamma = \frac{5}{3} \ , \quad v_c = 1 \ , \quad v_l = v_r = 0 \ , \quad c_l = c_r = 1 \ . \tag{1.22}$$

What are the numerical values of the ratios p_c/p_l and p_c/p_r of the pressure p_c at the contact interface to the pressures p_l and p_r at the left and right boundaries in this case?

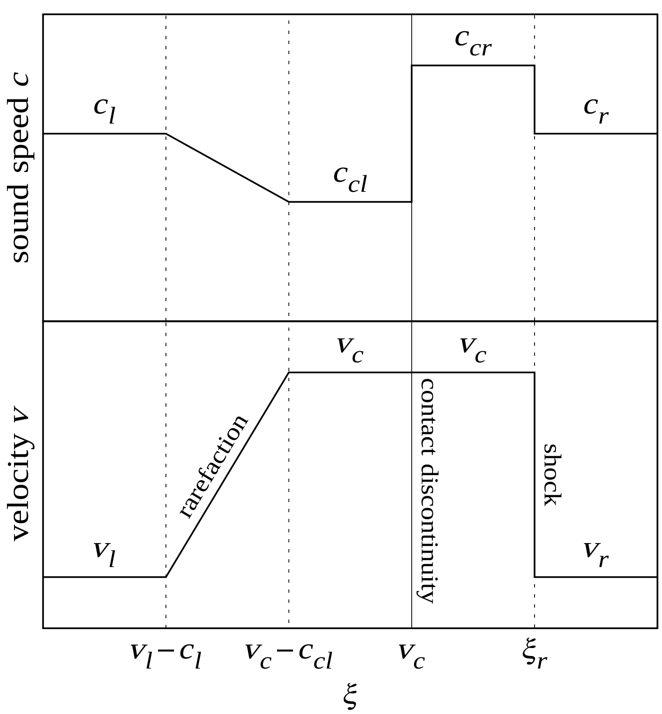

Figure 1: Structure of the solution to the Riemann problem, showing a rarefaction wave to the left and a shock wave to the right.