

1. 1D Lagrangian shock-capturing hydrodynamic code

Write a 1D Lagrangian shock-capturing hydrodynamic code for a fluid with adiabatic index γ . Take periodic boundary conditions; that is, identify the left edge of the first zone with the right edge of the last zone.

(a) Integrator

Implement the following Lagrangian scheme for integrating the 1D fluid equations for the position x , velocity v , and specific energy ε one timestep Δt :

$$\Delta x_{i+\frac{1}{2}} = \Delta t v_{i+\frac{1}{2}}, \quad (1.1a)$$

$$\Delta v_i = -\frac{\Delta t}{m} (p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}}), \quad (1.1b)$$

$$\Delta \varepsilon_i = -\frac{\Delta t}{m} (p_{i+\frac{1}{2}} v_{i+\frac{1}{2}} - p_{i-\frac{1}{2}} v_{i-\frac{1}{2}}). \quad (1.1c)$$

Here i labels centers of zones, while $i \mp \frac{1}{2}$ label the left and right edges of zones. The zone mass m is constant.

(b) Zone functions

Implement functions giving derived properties of zones given the primary properties of position x , velocity v , and specific energy ε :

1. Positions of zone centers x_i given zone edges $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$. [Hint: The zone centers x_i are used only for plotting, so the average $\frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$ is fine. To deal with periodic boundary conditions, code this as $x_{i-\frac{1}{2}} + \frac{1}{2} \bmod(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \text{boxsize})$.]
2. Zone densities $\rho_i = m/(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})$. [Hint: Again, mod the difference $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ with the boxsize.]
3. Zone pressures p_i given zone velocities v_i , densities ρ_i , and specific energies ε_i . [Hint: For a fluid of adiabatic index γ , the specific energy is $\varepsilon = \frac{1}{2}v^2 + \frac{1}{\gamma-1}(p/\rho)$.]
4. Zone sound speeds c_i given zone densities and pressures ρ_i and p_i . [Hint: $c^2 = \gamma(p/\rho)$.]

(c) Riemann solver

Implement a Riemann solver that yields the pressure $p_{i+\frac{1}{2}}$ and velocity $v_{i+\frac{1}{2}}$ at the contact interface between two zones having velocity, sound speed, and pressure $\{v_i, c_i, p_i\}$ on the left, and $\{v_{i+1}, c_{i+1}, p_{i+1}\}$ on the right. In Problem Set 10, you found that, with the notation

$$i \rightarrow l, \quad i + \frac{1}{2} \rightarrow c, \quad i + 1 \rightarrow r, \quad (1.2)$$

the solution of the Riemann problem is the implicit equation

$$p_c = p_l f(z_l) = p_r f(z_r) , \quad (1.3)$$

where

$$z_l \equiv \frac{v_l - v_c}{c_l} , \quad z_r \equiv \frac{v_c - v_r}{c_r} , \quad (1.4)$$

the function $f(z)$ being

$$f(z) = \begin{cases} \left(1 + \frac{\gamma - 1}{2} z\right)^{2\gamma/(\gamma-1)} & (z \leq 0) \text{ rarefaction ,} \\ 1 + \gamma z M & (z \geq 0) \text{ shock ,} \end{cases} \quad (1.5)$$

with M the Mach number

$$M = \frac{\gamma + 1}{4} z + \sqrt{\left(\frac{\gamma + 1}{4} z\right)^2 + 1} . \quad (1.6)$$

(d) Global parameters

Set the adiabatic index to be $\gamma = \frac{5}{3}$. Set the zone mass m to be some constant, say 1 for simplicity. Set the Courant parameter \mathcal{C} for a function giving the timestep Δt

$$\Delta t = \mathcal{C} \min \left(\frac{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}{c_i} \right) . \quad (1.7)$$

[Hint: Remember to mod $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ with the boxsize.] Choose a suitable number N of zones. Choose a suitable boxsize, say N for simplicity.

(e) Initial conditions

Set up the initial conditions as a top hat in density, with uniform zero velocity, and uniform sound speed.

(f) Hydro code

Write a driver routine that runs the code. After setting the initial conditions, the driver should repeatedly:

1. Call zone functions;
2. Call the Riemann solver;
3. Record variables for later plotting;
4. Set the timestep Δt ;
5. Call the integrator.

(g) Plot

Did it work? Does it look right?