ASTR 5540 Math Meth Fall 2008. Problem Set 3. Due Mon Sep 15

1. Hydrogen in fast shocks

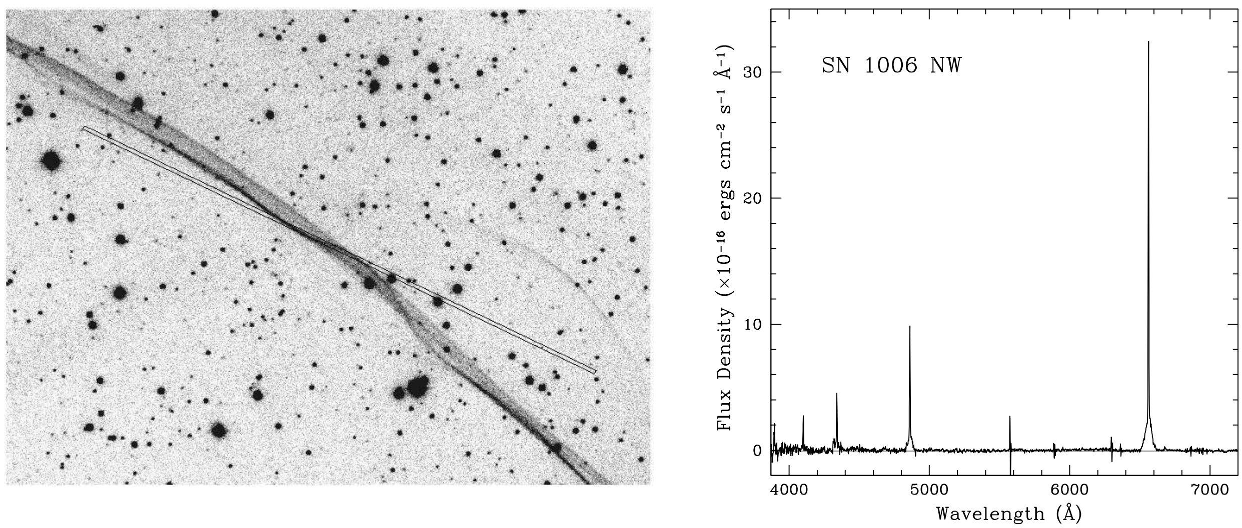

Figure 1: Hα emission from the northwest filament of Supernova 1006 (Ghavamian et al. 2002, http://arxiv.org/abs/astro-ph/0202487).

In a fast ($\gtrsim 1{,}000\,\mathrm{km\,s^{-1}}$) supernova shock such as observed in SN1006 or SN1987A, hydrogen is in three species, "slow" neutrals, "fast" neutrals, and protons. The slow and fast neutrals are observed through narrow and broad components to Lyα UV and Hα optical line emission, as illustrated for example in Figure 1. Because the ambient density is so low, the neutrals are almost entirely in the ground atomic state. Excitation from the ground state by collisions with electrons produces the observed line emission. The shock is collisionless, mediated by electromagnetic fields. Neutrals are not accelerated by the shock, so initially neutral hydrogen atoms entering the shocked zone remain cold and "slow". Neutrals are ionized to protons by electron impact. Being charged, the protons are immediately accelerated and heated, becoming "fast". Charge exchange between neutrals and protons, in which a neutral exchanges its electron with a proton, produces a population of fast neutrals. Because the temperature is so high, recombination from protons to neutrals is practically negligible.

(a) Solve

The number densities n_s of slow neutrals, n_f of fast neutrals, and n_p of protons, are governed by the system of equations

$$\begin{aligned} \frac{dn_s}{dt} &= -\gamma n_e n_s - \beta n_p n_s \ , \\ \frac{dn_f}{dt} &= -\gamma n_e n_f + \beta n_p n_s \ , \\ \frac{dn_p}{dt} &= n_e \gamma (n_s + n_f) \ , \end{aligned} \tag{1.1}$$

where n_e is the number density of electrons, which by charge neutrality satisfies

$$n_e = n_p \ . \tag{1.2}$$

The coefficient γ is the ionization rate, while β is the charge exchange rate. Define an "ionization time" τ by

$$d\tau = n_e \gamma \, dt \tag{1.3}$$

so that equations (1.1) become

$$\begin{aligned} \frac{dn_s}{d\tau} &= -n_s - (\beta/\gamma) n_s \ , \\ \frac{dn_f}{d\tau} &= -n_f + (\beta/\gamma) n_s \ , \\ \frac{dn_p}{d\tau} &= n_s + n_f \ . \end{aligned} \tag{1.4}$$

Write equations (1.4) as a matrix equation $d\boldsymbol{n}/d\tau = \boldsymbol{A}\boldsymbol{n}$. Solve it analytically, taking the rate coefficients γ and β to be constants. Express your solution for each component n_i as a function of the initial values $n_i(0)$ and the collision time τ. [Hint: You are welcome to use a program such as Mathematica to solve the equations, if you wish. Be aware that Mathematica's Eigenvectors[] function returns eigenvectors as rows, not columns, so you will probably want to take the transpose of the matrix of eigenvectors. In other words, $\boldsymbol{P} =$ Transpose[Eigenvectors[$\boldsymbol{A}$]] solves $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}$ where $\boldsymbol{\Lambda} =$ DiagonalMatrix[Eigenvalues[$\boldsymbol{A}$]] is the diagonal matrix of eigenvalues.]

(b) Comment

Comment on the eigenvalues: Are there any zero eigenvalues? Negative eigenvalues? Positive eigenvalues? Does this make sense? If you wanted to solve the system of equations numerically, would it be a good idea to use a stiff integration technique?

(c) Plot

Ionization and charge exchange rates may be obtained from the International Atomic Energy Agency's Aladdin database (`http://www-amdis.iaea.org/ALADDIN/`). At the electron and proton temperatures $T_e \approx 3\,\text{keV}$ and $T_p \approx 50\,\text{keV}$ inferred in SN1006, the rate coefficients are approximately $\gamma \approx 1.5 \times 10^{-14}\,\text{m}^3\,\text{s}^{-1}$ and $\beta \approx 3 \times 10^{-14}\,\text{m}^3\,\text{s}^{-1}$. Plot the relative number densities n_i/n_{tot}, where $n_{\text{tot}} = n_s + n_f + n_p$ is the total hydrogen density, as a function of ionization time τ, for the case of equal initial neutral and ionized fractions, $n_s(0) = n_p(0)$. Note that initially there are no fast neutrals, $n_f(0) = 0$.