

ASTR 5540 Math Meth Fall 2008. Problem Set 5. Due Mon Sep 29

1. Forced, damped, simple harmonic oscillator

The problem is to solve the forced, damped, simple harmonic oscillator (overdot signifies differentiation d/dt with respect to time)

$$\ddot{z} + 2a\dot{z} + k^2z = f(t) , \quad (1.1)$$

subject to a sinusoidal forcing function that turns on at $t = 0$,

$$f(t) = \begin{cases} 0 & (t \leq 0) \\ c \sin(\omega t) & (t \geq 0) . \end{cases} \quad (1.2)$$

Here a , k , c , and ω are all constants. The forced, damped, simple harmonic oscillator equation is a prototype for equations that appear ubiquitously in the physical sciences, wherever there are small amplitude perturbations, such as waves of any kind, that have a natural frequency k , are damped with damping constant a , and are excited by some external forcing function.

(a) Homogeneous solutions

Find the homogeneous solutions to equation (1.1). Consider cases $a < k$, $a > k$, and $a = k$. In the last case, $a = k$, use the Wronskian to find a second solution. [Hint: Assume without loss of generality that k is positive. You may also assume that a is positive. Why is the damping constant a likely to be positive in physical situations? I found it convenient to introduce a quantity $q \equiv \sqrt{k^2 - a^2}$.]

(b) Green's function

Find the Green's function $G(t, t_0)$ of the damped simple harmonic oscillator, the solution to

$$\ddot{G} + 2a\dot{G} + k^2G = \delta_D(t - t_0) \quad (1.3)$$

subject to the “retarded” boundary condition that $G(t, t_0)$ vanishes for $t < t_0$.

(c) Solution for oscillating forcing function

Use the Green's function to solve equation (1.1) subject to the forcing function (1.2).

(d) Plot and comment

Plot and comment on a representative sample of solutions with various a , k , c , and ω . Show that, after a long time, the solution oscillates with amplitude

$$\frac{c}{\sqrt{(k^2 - \omega^2)^2 + (2a\omega)^2}} \quad (1.4)$$

What happens “near resonance”, whatever that means? [Hint: In thinking about solutions with various a , k , c , and ω , you can scale to $k = 1$ and $c = 1$ without loss of generality. Why?]