ASTR 5540 Math Meth Fall 2008. Problem Set 6. Due Mon Oct 6

1. Spherical Harmonics

Spherical harmonics $Y_{\ell m}(\theta, \phi)$, the eigenmodes of the angular momentum (or rotation) operator $\boldsymbol{L}$, constitute the second most important complete orthonormal set of functions, the most important being the Fourier modes e^{ikr}, the eigenmodes of the momentum (or translation) operator $-i\boldsymbol{\nabla}$. Spherical harmonics crop up wherever there is rotational symmetry — the electronic wave functions of atoms, the oscillations of spherical objects such as planets and stars, the Cosmic Microwave Background, and many more.

Let θ, ϕ be polar coordinates, with θ the polar angle and ϕ the azimuthal angle about the vertical direction $\boldsymbol{z}$. It can be shown that the square of the angular momentum operator is expressible in the form

$$L^2 = -\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \ . \tag{1.1}$$

(a) Separate variables

By separating variables, $\psi(\theta, \phi) = \Theta(\theta)\Phi(\phi)$, show that the eigenvalue equation

$$L^2\psi(\theta, \phi) = \ell(\ell + 1)\psi(\theta, \phi) \tag{1.2}$$

leads to ordinary differential equations for Φ and Θ

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi \ , \tag{1.3}$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} - \frac{m^2}{\sin^2\theta}\Theta = -\ell(\ell + 1)\Theta \ . \tag{1.4}$$

(b) Solution for Φ, and admissible values of m

What are the solutions of the eigenvalue equation (1.3) for $\Phi(\phi)$, subject to the periodicity constraint $\Phi(\phi + 2\pi) = \Phi(\phi)$? What are the possible values of the eigenvalues m?

(c) Associated Legendre equation

One wants to define orthogonality of spherical harmonics with respect to integrals over solid angle. An element of spherical angle is $do = \sin\theta\, d\theta d\phi = dx d\phi$ where $x \equiv \cos\theta$ (the minus sign disappears provided that the integrals over x are taken from -1 to 1 rather than from 1 to -1). For the purpose of expressing the differential equation for Θ in Sturm-Liouville form, it is therefore simplest to recast it in terms of x. Show that the differential equation (1.4) for Θ in terms of x is the **associated Legendre equation**

$$(1 - x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} - \frac{m^2}{1 - x^2}\Theta = -\ell(\ell + 1)\Theta \ . \tag{1.5}$$

Is this equation (1.5) in Sturm-Liouville form?

(d) Series solution for Θ

Find series solutions of the form

$$\Theta(x) = (1 - x^2)^{m/2} \sum_{n}^{\infty} a_k x^k \tag{1.6}$$

for the associated Legendre equation (1.5). Verify that the indicial equation is

$$n(n-1) = 0 \; . \tag{1.7}$$

How many distinct series solutions are there? [Hint: Make the substitution $\Theta = (1-x^2)^{m/2}\chi$ to transform the associated Legendre equation to

$$(1 - x^2)\frac{d^2\chi}{dx^2} - 2(m+1)x\frac{d\chi}{dx} + (\ell - m)(\ell + m + 1)\chi = 0 \tag{1.8}$$

and then try a series expansion of χ.]

(e) Admissible values of ℓ

For what values of ℓ does your series in part (d) terminate? These polynomial solutions are the Associated Legendre polynomials. [It is straightforward to confirm (but I'm not asking you to do this) that if the series does not terminate, then it blows up at $x = \pm 1$. Therefore the physically interesting solutions, those which are finite at $x = \pm 1$, are precisely those for which the series terminates.]