ASTR 5540 Math Meth Fall 2008. Problem Set 7. Due Mon Oct 13

1. Numerical computation of spherical harmonics

For positive m, the orthonormal spherical harmonics $Y_{lm}(\theta, \phi)$ are given explicitly in terms of associated Legendre polynomials $P_l^m(x)$ by

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \quad (m = 0, ..., l) \ . \tag{1.1}$$

For negative m,

$$Y_{l,-m}(\theta, \phi) = (-)^m Y^*_{lm}(\theta, \phi) \ . \tag{1.2}$$

Usually, the fastest and most accurate way to compute the associated Legendre polynomials $P_l^m(x)$ is to use recurrence relations. This is especially true if you want not just one spherical harmonic but a full set of them, complete up to some maximum harmonic number $l_{\max}$. The associated Legendre polynomials satisfy many recurrence relations, but the numerically useful one proves to be

$$(l - m + 1) P_{l+1}^m(x) = (2l + 1) x P_l^m(x) - (l + m) P_{l-1}^m(x) \ . \tag{1.3}$$

If the recursion is applied in the direction of increasing l, then the starting point is

$$P_l^l(x) = (-)^l (2l - 1)!! (1 - x^2)^{l/2} \tag{1.4}$$

and of course $P_{l-1}^l(x) = 0$.

(a) Stability

Investigate analytically the stability of the recurrence relation (1.3). Show that the stability is neutral if $|x|$ is not near 1, but that there are growing and decaying modes if $|x|$ is near 1. Given that the leading term in the series of expansion of $P_l^m(x)$ near $|x| = 1$ is

$$P_l^m(x) = (-)^m \frac{(l+m)!}{(l-m)!} \left(\frac{1 - x^2}{4} \right)^{m/2} \tag{1.5}$$

in which direction would you expect the recurrence to be stable, increasing or decreasing l?

(b) Code

In the language of your choice (mathematica, IDL, c, ...), write code that implements the recurrence relation (1.3). The code should take as inputs $l_{\max}$ and x, and it should return a set of $P_l^m(x)$ complete up to harmonic number $l_{\max}$.

(c) Plot and comment

Plot a selection of $P_l^m(x)$ for suitably large l, say $l \approx 100$. Comment.