

2 Fundamentals of General Relativity

2.1 Concept Questions

1. What assumption of GR makes it possible to introduce a coordinate system?
2. Is the speed of light a universal constant in GR? If so, in what sense?
3. What does “locally inertial” mean? How local is local?
4. Why is spacetime locally inertial?
5. What assumption of GR makes it possible to introduce clocks and rulers?
6. Consider two observers at the same point and with the same instantaneous velocity, but one is accelerating and the other is in free-fall. What is the relation between the proper time or proper distance along an infinitesimal interval measured by the two observers? What assumption of GR implies this?
7. Does the (Strong) Principle of Equivalence imply that two unequal masses will fall at the same rate in a gravitational field? Explain.
8. In what respects is the Strong Principle of Equivalence (gravity is equivalent to acceleration) stronger than the Weak Principle of Equivalence (gravitating mass equals inertial mass)?
9. Standing on the surface of the Earth, you hold an object of negative mass in your hand, and drop it. According to the Principle of Equivalence, does the negative mass fall up or down?
10. Same as the previous question, but what does Newtonian gravity predict?
11. You have a box of negative mass particles, and you remove energy from it. Do the particles move faster or slower? Does the entropy of the box increase or decrease? Does the pressure exerted by the particles on the walls of the box increase or decrease?
12. You shine two light beams along identical directions in a gravitational field. The two light beams are identical in every way except that they have two different frequencies. Does the Equivalence Principle imply that the interference pattern produced by each of the beams individually is the same?
13. What is a “straight line”, according to the Principle of Equivalence?
14. If all objects move on straight lines, how is it that when, standing on the surface of the Earth, you throw two objects in the same direction but with different velocities, they follow two different trajectories?
15. In relativity, what is the generalization of the “shortest distance between two points”?

16. What kinds of general coordinate transformations are allowed in GR?
17. In GR, what is a scalar? A 4-vector? A tensor? Which of the following is a scalar/vector/tensor/none-of-the-above? (a) a set of coordinates x^μ ; (b) a coordinate interval dx^μ ; (c) proper time τ ?
18. What does general covariance mean?
19. Why is it important to define covariant derivatives that behave like tensors?
20. Is covariant differentiation a derivation? That is, is covariant differentiation a linear operation, and does it obey the Leibniz rule for the derivative of a product?
21. What is the covariant derivative of the metric tensor? Explain.
22. What does a connection coefficient $\Gamma_{\mu\nu}^\kappa$ mean physically? Is it a tensor? Why, or why not?
23. An astronaut is in free-fall in orbit around the Earth. Can the astronaut detect that there is a gravitational field?
24. Can a gravitational field exist in flat space?
25. How can you tell whether a given metric is equivalent to the Minkowski metric of flat space?
26. How many degrees of freedom does the metric have? How many of these degrees of freedom can be removed by arbitrary transformations of the spacetime coordinates, and therefore how many physical degrees of freedom are there in spacetime?
27. If you insist that the spacetime is spherical, how many physical degrees of freedom are there in the spacetime?
28. If you insist that the spacetime is spatially homogeneous and isotropic (the cosmological principle), how many physical degrees of freedom are there in the spacetime?
29. In GR, you are free to prescribe any spacetime (any metric) you like, including metrics with wormholes and metrics that connect the future to the past so as to violate causality. True or false?
30. If it is true that in GR you can prescribe any metric you like, then why aren't you bumping into wormholes and causality violations all the time?
31. How much mass does it take to curve space significantly?
32. What is the relation between the energy-momentum 4-vector of a particle and the energy-momentum tensor?

33. It is straightforward to go from a prescribed metric to the energy-momentum tensor. True or false?
34. It is straightforward to go from a prescribed energy-momentum tensor to the metric. True or false?
35. Does the Principle of Equivalence imply Einstein's equations?
36. What do Einstein's equations mean physically?
37. What does the Riemann curvature tensor $R_{\kappa\lambda\mu\nu}$ mean physically? Is it a tensor?
38. The Riemann tensor splits into compressive (Ricci) and tidal (Weyl) parts. What do these parts mean, physically?
39. Einstein's equations imply conservation of energy-momentum, but what does that mean?
40. Do Einstein's equations describe gravitational waves?
41. Do photons (massless particles) gravitate?
42. How do different forms of mass-energy gravitate?
43. How does negative mass gravitate?

2.2 What's important?

This section of the notes adopts the traditional coordinate-based approach to GR. The approach is neither the most insightful nor the most powerful, but it is the fastest route to connecting the metric to the energy-momentum content of spacetime.

1. Postulates of GR. How do the various postulates imply the mathematical structure of GR?
2. The road from spacetime curvature to energy-momentum:
metric $g_{\mu\nu}$
→ connection coefficients $\Gamma_{\mu\nu}^{\kappa}$
→ Riemann curvature tensor $R_{\kappa\lambda\mu\nu}$
→ Ricci tensor $R_{\kappa\mu}$ and scalar R
→ Einstein tensor $G_{\kappa\mu} = R_{\kappa\mu} - \frac{1}{2}g_{\kappa\mu}R$
→ energy-momentum tensor $T_{\kappa\mu}$
3. 4-velocity and 4-momentum. Geodesic equation.
4. Bianchi identities guarantee conservation of energy-momentum.

2.3 The postulates of General Relativity

1. Spacetime is a 4-dimensional manifold
2. The (Strong) Principle of Equivalence
3. Einstein's Equations
4. No torsion

2.3.1 Spacetime is a 4-dimensional manifold

A 4-dimensional manifold is defined mathematically to be a topological space that is locally homomorphic to Euclidean 4-space \mathbb{R}^4 .

This postulate implies that it is possible to set up a coordinate system (possibly in patches)

$$x^\mu \equiv \{x^0, x^1, x^2, x^3\} \quad (1)$$

such that each point of (the patch of) spacetime has a unique coordinate.

Andrew's convention:

Greek dummy indices label curved spacetime coordinates.

Latin dummy indices label locally inertial coordinates.

2.3.2 (Strong) Principle of Equivalence (PE)

"The laws of physics in a gravitating frame are equivalent to those in an accelerating frame".

The Weak Principle of Equivalence is "Gravitating mass = inertial mass".

PE \Rightarrow spacetime is **locally inertial** (see §2.4).

2.3.3 Einstein's equations

Einstein's equations comprise a 4×4 symmetric matrix of equations

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \quad (2)$$

G is the Newtonian gravitational constant.

$G_{\mu\nu}$ is the **Einstein tensor**.

$T_{\mu\nu}$ is the **energy-momentum tensor**.

Physically

$$\text{(compressive part of) curvature} = \text{energy-momentum content} \quad (3)$$

Generalizes Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho \quad (4)$$

where Φ is the Newtonian gravitational potential, and ρ the mass-energy density. Poisson's equation is the time-time component of Einstein's equations in the limit of a weak gravitational field and slowly moving matter.

2.4 Existence of locally inertial frames

PE \Rightarrow

at each point of spacetime it is possible to choose an inertial, or free-fall, frame (i.e. a set of coordinates) in which the laws of special relativity apply within an infinitesimal interval of that point, that is

- objects move in straight lines at constant velocity in the absence of electromagnetic and other forces;
- the speed of light is a universal constant.

It is built into this postulate that GR is, like special relativity, a **metric theory**. Notably, the proper times and distances measured by an accelerating observer are the same as those measured by a freely-falling observer at the same point and with the same instantaneous velocity.

2.5 Metric

The metric is the essential mathematical object that converts an infinitesimal coordinate interval

$$dx^\mu \equiv \{dx^0, dx^1, dx^2, dx^3\} \quad (5)$$

to a proper measurement of an interval of time or space.

Postulate 1 \Rightarrow can choose coordinates

$$x^\mu \equiv \{x^0, x^1, x^2, x^3\} \quad (6)$$

covering (a patch of) spacetime.

Postulate 2 \Rightarrow at each point of spacetime can choose locally inertial coordinates

$$\xi^m \equiv \{\xi^0, \xi^1, \xi^2, \xi^3\} \quad (7)$$

such that

$$ds^2 = \eta_{mn} d\xi^m d\xi^n \quad (8)$$

in an infinitesimal neighborhood of the point. Here η_{mn} is the Minkowski metric of SR.

Since

$$d\xi^m = \frac{\partial \xi^m}{\partial x^\mu} dx^\mu \quad (9)$$

it follows that

$$ds^2 = \eta_{mn} \frac{\partial \xi^m}{\partial x^\mu} \frac{\partial \xi^n}{\partial x^\nu} dx^\mu dx^\nu \quad (10)$$

so

$$\boxed{ds^2 = g_{\mu\nu} dx^\mu dx^\nu} \quad (11)$$

where $g_{\mu\nu}$ is the **metric**, a 4×4 symmetric matrix

$$g_{\mu\nu} = \eta_{mn} \frac{\partial \xi^m}{\partial x^\mu} \frac{\partial \xi^n}{\partial x^\nu} \quad (12)$$

2.6 Basis \mathbf{g}_μ of tangent vectors

You are familiar with the idea that in ordinary 3D Euclidean geometry it is often convenient to treat vectors in an abstract coordinate-independent formalism. Thus for example a 3-vector is commonly written as an abstract quantity \mathbf{r} . The coordinates of the vector \mathbf{r} may be $\{x, y, z\}$ in some particular coordinate system, but one recognizes that the vector has a meaning, a magnitude and a direction, that is independent of the coordinate system adopted. In an arbitrary Cartesian coordinate system, the Euclidean 3-vector \mathbf{r} can be expressed

$$\mathbf{r} = \sum_i \hat{\mathbf{x}}_i x_i = \hat{\mathbf{x}} x + \hat{\mathbf{y}} y + \hat{\mathbf{z}} z \quad (13)$$

where $\hat{\mathbf{x}}_i \equiv \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ are unit vectors along each of the coordinate axes.

The same kind of abstract notation is useful in GR. Define \mathbf{g}_μ

$$\mathbf{g}_\mu \equiv \{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\} \quad (14)$$

to be the basis of axes tangent to the coordinates x^μ . Each axis \mathbf{g}_μ is a 4D vector object, with both magnitude and direction in spacetime.

An interval dx^μ of spacetime can be expressed in coordinate-independent fashion as the vector interval $d\mathbf{x}$

$$d\mathbf{x} \equiv \mathbf{g}_\mu dx^\mu = \mathbf{g}_0 dx^0 + \mathbf{g}_1 dx^1 + \mathbf{g}_2 dx^2 + \mathbf{g}_3 dx^3 . \quad (15)$$

At this point, the notation $d\mathbf{x}$ is merely symbolic: $d\mathbf{x}$ is not necessarily a total differential of something. However, GR assumes the no-torsion condition that $d\mathbf{x}$ is in fact the total differential $d\mathbf{X}$ of some position vector \mathbf{X}

$$d\mathbf{x} = d\mathbf{X} . \quad (16)$$

The physical meaning of torsion is discussed in §2.39. If the no-torsion condition (16) is true, then equation (15) shows that \mathbf{g}_μ are partial derivatives of the position vector \mathbf{X} :

$$\mathbf{g}_\mu = \frac{\partial \mathbf{X}}{\partial x^\mu} . \quad (17)$$

The scalar length squared of the vector interval $d\mathbf{x}$ is

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu dx^\mu dx^\nu \quad (18)$$

whence

$$\boxed{g_{\mu\nu} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu} \quad (19)$$

the metric is the (4D) scalar product of tangent vectors.

2.7 Contravariant 4-vector

Under a general coordinate transformation

$$x^\mu \rightarrow x'^\mu \quad (20)$$

a coordinate interval dx^μ transforms as

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu . \quad (21)$$

In GR, a **4-vector** is defined to be a quantity $A^\mu = \{A^0, A^1, A^2, A^3\}$ that transforms under a coordinate transformation (20) like a coordinate interval

$$\boxed{A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu} . \quad (22)$$

A 4-vector may be written in coordinate-independent fashion as

$$\mathbf{A} = \mathbf{g}_\mu A^\mu . \quad (23)$$

2.8 Lowering and raising indices

Define $g^{\mu\nu}$ to be the **inverse metric**, satisfying

$$g_{\lambda\mu} g^{\mu\nu} = \delta_\lambda^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \quad (24)$$

The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ provide the means of lowering and raising indices. The components of a 4-vector A^μ with raised index are called its **contravariant** components, while those A_μ with lowered indices are called its **covariant** components,

$$\boxed{A_\mu = g_{\mu\nu} A^\nu} , \quad (25)$$

$$\boxed{A^\mu = g^{\mu\nu} A_\nu} . \quad (26)$$

2.9 Covariant 4-vector

Under a general coordinate transformation (20), the covariant components A_μ of a 4-vector transform as

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu . \quad (27)$$

You can check that the transformation law (27) for the covariant components A_μ is consistent with the transformation law (22) for the contravariant components A^μ .

You can check that the tangent vectors \mathbf{g}_μ transform as a covariant 4-vector.

2.10 Scalar product

If A^μ and B^μ are 4-vectors, then their scalar product is

$$A_\mu B^\mu = A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu . \quad (28)$$

This is a scalar, a quantity that remains invariant under general coordinate transformations.

In abstract vector formalism, the scalar product of two 4-vectors $\mathbf{A} = \mathbf{g}_\mu A^\mu$ and $\mathbf{B} = \mathbf{g}_\mu B^\mu$ is

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu A^\mu B^\nu = g_{\mu\nu} A^\mu B^\nu . \quad (29)$$

2.11 Tensor

In general, a **tensor** $A_{\mu\nu\dots}^{\kappa\lambda\dots}$ is an object that transforms under general coordinate transformations (20) as

$$\boxed{A'_{\mu\nu\dots}{}^{\kappa\lambda\dots} = \frac{\partial x'^\kappa}{\partial x^\alpha} \frac{\partial x'^\lambda}{\partial x^\beta} \dots \frac{\partial x^\gamma}{\partial x'^\mu} \frac{\partial x^\delta}{\partial x'^\nu} \dots A_{\gamma\delta\dots}{}^{\alpha\beta\dots}} . \quad (30)$$

You can check that the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are indeed tensors, transforming like (30).

The **rank** of a tensor is the number of indices.

A scalar is a tensor of rank 0.

A 4-vector is a tensor of rank 1.

2.12 Derivative of a scalar

Suppose that Φ is a scalar. Then the coordinate derivative of Φ is a 4-vector

$$\boxed{\frac{\partial \Phi}{\partial x^\mu}} \text{ is a covariant 4-vector} \quad (31)$$

transforming like equation (27).

As a shorthand, the ordinary partial derivative is often denoted with a comma

$$\frac{\partial \Phi}{\partial x^\mu} = \Phi_{,\mu} . \quad (32)$$

2.13 Derivative of a 4-vector

The ordinary partial derivative of a covariant 4-vector A^μ is not a tensor

$$\frac{\partial A^\mu}{\partial x^\nu} \text{ is not a tensor} \quad (33)$$

because it does not transform like a tensor.

However, the 4-vector $\mathbf{A} = \mathbf{g}_\mu A^\mu$, being by construction invariant under coordinate transformations, is a scalar, and its partial derivative is a 4-vector

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial x^\nu} &= \frac{\partial \mathbf{g}_\mu A^\mu}{\partial x^\nu} \\ &= \mathbf{g}_\mu \frac{\partial A^\mu}{\partial x^\nu} + \frac{\partial \mathbf{g}_\mu}{\partial x^\nu} A^\mu \quad \text{is a tensor .}\end{aligned}\tag{34}$$

2.14 Connection coefficients (Christoffel symbols)

The expression on the right hand side of equation (34) involves the derivatives of the basis vectors \mathbf{g}_μ . These derivatives define the **connection coefficients** $\Gamma_{\mu\nu}^\kappa$, also known as Christoffel symbols,

$$\boxed{\frac{\partial \mathbf{g}_\mu}{\partial x^\nu} \equiv \Gamma_{\mu\nu}^\kappa \mathbf{g}_\kappa} \quad \text{is not a tensor .}\tag{35}$$

2.15 Covariant derivative of a contravariant 4-vector

Expression (34) along with the definition (35) of the connection coefficients implies that

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial x^\nu} &= \mathbf{g}_\mu \frac{\partial A^\mu}{\partial x^\nu} + \Gamma_{\mu\nu}^\kappa \mathbf{g}_\kappa A^\mu \\ &= \mathbf{g}_\kappa \left(\frac{\partial A^\kappa}{\partial x^\nu} + \Gamma_{\mu\nu}^\kappa A^\mu \right) \quad \text{is a tensor .}\end{aligned}\tag{36}$$

The expression in parentheses is a tensor, and defines the **covariant derivative** $D_\nu A^\kappa$ of the contravariant 4-vector A^κ

$$\boxed{D_\nu A^\kappa \equiv \frac{\partial A^\kappa}{\partial x^\nu} + \Gamma_{\mu\nu}^\kappa A^\mu} \quad \text{is a tensor .}\tag{37}$$

As a shorthand, the covariant derivative is often denoted with a semi-colon

$$D_\nu A^\kappa = A^\kappa_{;\nu} .\tag{38}$$

2.16 Covariant derivative of a covariant 4-vector

Similarly,

$$\frac{\partial \mathbf{A}}{\partial x^\nu} = \mathbf{g}^\kappa D_\nu A_\kappa \quad \text{is a tensor}\tag{39}$$

where $D_\nu A_\kappa$ is the covariant derivative of the covariant 4-vector A_κ

$$\boxed{D_\nu A_\kappa \equiv \frac{\partial A_\kappa}{\partial x^\nu} - \Gamma_{\kappa\nu}^\mu A_\mu} \quad \text{is a tensor .}\tag{40}$$

2.17 Covariant derivative of a tensor

In general, the covariant derivative of a tensor is

$$D_\alpha A_{\mu\nu\dots}^{\kappa\lambda\dots} = \frac{\partial A_{\mu\nu\dots}^{\kappa\lambda\dots}}{\partial x^\alpha} + \Gamma_{\beta\alpha}^\kappa A_{\mu\nu\dots}^{\beta\lambda\dots} + \Gamma_{\beta\alpha}^\lambda A_{\mu\nu\dots}^{\kappa\beta\dots} + \dots - \Gamma_{\mu\alpha}^\beta A_{\beta\nu\dots}^{\kappa\lambda\dots} - \Gamma_{\nu\alpha}^\beta A_{\mu\beta\dots}^{\kappa\lambda\dots} - \dots \quad (41)$$

with a positive Γ term for each contravariant index, and a negative Γ term for each covariant index.

2.18 No-torsion condition

Recall that the no-torsion condition (16) implies that $\mathbf{g}_\mu = \partial\mathbf{X}/\partial x^\mu$, equation (17). It follows that

$$\Gamma_{\mu\nu}^\kappa \mathbf{g}_\kappa \equiv \frac{\partial \mathbf{g}_\mu}{\partial x^\nu} = \frac{\partial^2 \mathbf{X}}{\partial x^\nu \partial x^\mu} \quad (42)$$

is symmetric in $\mu\nu$, because of the commutation of partial derivatives. This implies the no-torsion condition that the connection coefficients are symmetric in their last two indices

$$\boxed{\Gamma_{\mu\nu}^\kappa = \Gamma_{\nu\mu}^\kappa} . \quad (43)$$

The no-torsion condition (16) implies the symmetry (43), and the converse is also true. For, from the definitions (35) of the connection coefficients, the symmetry (43) implies that

$$\frac{\partial \mathbf{g}_\mu}{\partial x^\nu} - \frac{\partial \mathbf{g}_\nu}{\partial x^\mu} = 0 \quad (44)$$

which, according to Frobenius' theorem, is precisely the condition for the system \mathbf{g}_μ to be integrable, that is, there exists a position vector \mathbf{X} whose partial derivatives are $\partial\mathbf{X}/\partial x^\mu = \mathbf{g}_\mu$, equation (17), and then the no-torsion condition (16) is true.

2.19 Torsion tensor

GR assumes no torsion, but it is possible to consider generalizations to theories with torsion. The **torsion tensor** $S_{\kappa\lambda}^\mu$ is defined by the commutator of the covariant derivative acting on a scalar Φ

$$\boxed{[D_\kappa, D_\lambda] \Phi = S_{\kappa\lambda}^\mu \frac{\partial \Phi}{\partial x^\mu}} \quad \text{is a tensor} . \quad (45)$$

Note that the covariant derivative of a scalar is just the ordinary derivative, $D_\lambda \Phi = \partial\Phi/\partial x^\lambda$. The expression (40) for the covariant derivatives shows that the torsion tensor is

$$\boxed{S_{\kappa\lambda}^\mu = \Gamma_{\kappa\lambda}^\mu - \Gamma_{\lambda\kappa}^\mu} \quad \text{is a tensor} \quad (46)$$

which is evidently antisymmetric in the indices $\kappa\lambda$.

In Einstein-Cartan theory, the torsion tensor is related to the spin content of spacetime. Since this vanishes in empty space, Einstein-Cartan theory is indistinguishable from GR in experiments carried out in vacuum.

2.20 Connection coefficients in terms of the metric

$$\begin{aligned}
\frac{\partial g_{\lambda\mu}}{\partial x^\nu} &= \frac{\partial \mathbf{g}_\lambda \cdot \mathbf{g}_\mu}{\partial x^\nu} \\
&= \mathbf{g}_\lambda \cdot \frac{\partial \mathbf{g}_\mu}{\partial x^\nu} + \mathbf{g}_\mu \cdot \frac{\partial \mathbf{g}_\lambda}{\partial x^\nu} \\
&= \mathbf{g}_\lambda \cdot \mathbf{g}_\kappa \Gamma_{\mu\nu}^\kappa + \mathbf{g}_\mu \cdot \mathbf{g}_\kappa \Gamma_{\lambda\nu}^\kappa \\
&= g_{\lambda\kappa} \Gamma_{\mu\nu}^\kappa + g_{\mu\kappa} \Gamma_{\lambda\nu}^\kappa \\
&= \Gamma_{\lambda\mu\nu} + \Gamma_{\mu\lambda\nu}
\end{aligned} \tag{47}$$

where

$$\Gamma_{\lambda\mu\nu} \equiv g_{\lambda\kappa} \Gamma_{\mu\nu}^\kappa . \tag{48}$$

So

$$\begin{aligned}
\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \Gamma_{\lambda\mu\nu} + \Gamma_{\mu\lambda\nu} + \Gamma_{\lambda\nu\mu} + \Gamma_{\nu\lambda\mu} - \Gamma_{\mu\nu\lambda} - \Gamma_{\nu\mu\lambda} \\
&= 2\Gamma_{\lambda\mu\nu} - S_{\lambda\mu\nu} - S_{\mu\nu\lambda} - S_{\nu\mu\lambda} \\
&= 2\Gamma_{\lambda\mu\nu}
\end{aligned} \tag{49}$$

the last line of which follows from the no-torsion symmetry (16). Thus

$$\boxed{\Gamma_{\lambda\mu\nu} = \frac{1}{2} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right)} \text{ is not a tensor .} \tag{50}$$

This is the formula that allows connection coefficients to be calculated from the metric.

2.21 4-velocity

Consider a particle following a worldline

$$x^\mu(\tau) \tag{51}$$

where τ is the particle's proper time. The proper time along any interval of the worldline is $d\tau \equiv \sqrt{ds^2}$.

Define the **4-velocity** u^μ by

$$\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau}} \text{ is a 4-vector .} \tag{52}$$

The magnitude squared of the 4-velocity is constant

$$u_\mu u^\mu = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{ds^2}{d\tau^2} = 1 . \tag{53}$$

This can be regarded as an integral of motion associated with conservation of particle rest mass per unit mass.

2.22 Geodesic equation

Let $\mathbf{u} \equiv \mathbf{g}_\mu u^\mu$ be the 4-velocity in coordinate-independent notation. The Principle of Equivalence implies that the **geodesic equation**, the equation of motion of a freely-falling particle, is

$$\boxed{\frac{d\mathbf{u}}{d\tau} = 0} . \quad (54)$$

Why? Because $d\mathbf{u}/d\tau = 0$ in the particle's own free-fall frame, and the equation is coordinate-independent. In the particle's own free-fall frame, the particle's 4-velocity is $u^\mu = \{1, 0, 0, 0\}$, and the particle's locally inertial axes $\mathbf{g}_\mu = \{\mathbf{g}_t, \mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z\}$ are constant.

What does the equation of motion look like in coordinate notation? The acceleration is

$$\begin{aligned} \frac{d\mathbf{u}}{d\tau} &= \frac{dx^\nu}{d\tau} \frac{\partial \mathbf{u}}{\partial x^\nu} \\ &= u^\nu g_\kappa D_\nu u^\kappa \\ &= u^\nu g_\kappa \left(\frac{\partial u^\kappa}{\partial u^\nu} + \Gamma_{\mu\nu}^\kappa u^\mu \right) \\ &= g_\kappa \left(\frac{du^\kappa}{d\tau} + \Gamma_{\mu\nu}^\kappa u^\mu u^\nu \right) . \end{aligned} \quad (55)$$

The geodesic equation is then

$$\boxed{\frac{du^\kappa}{d\tau} + \Gamma_{\mu\nu}^\kappa u^\mu u^\nu = 0} . \quad (56)$$

Another way of writing the geodesic equation is

$$\frac{Du^\kappa}{D\tau} = 0 \quad (57)$$

where $D/D\tau$ is the covariant proper time derivative

$$\frac{D}{D\tau} \equiv u^\nu D_\nu . \quad (58)$$

2.23 4-momentum

The 4-momentum of a particle of rest mass m is defined to be

$$\boxed{p^\mu \equiv m u^\mu = m \frac{dx^\mu}{d\tau}} \text{ is a 4-vector} . \quad (59)$$

The momentum squared is

$$p_\mu p^\mu = m^2 u_\mu u^\mu = m^2 \quad (60)$$

the square of the rest mass.

2.24 Affine parameter

For photons, the rest mass is zero, $m = 0$, but the 4-momentum p^μ remains finite. Define the **affine parameter** λ by

$$\boxed{\lambda \equiv \frac{\tau}{m}} \quad \text{is a scalar} \quad (61)$$

which remains finite in the limit $m \rightarrow 0$. The affine parameter λ is unique up to an overall linear transformation (that is, $\alpha\lambda + \beta$ is also an affine parameter, for constant α and β), because of the freedom in the choice of mass m and the zero point of proper time τ . In terms of the affine parameter, the 4-momentum is

$$\boxed{p^\mu = \frac{dx^\mu}{d\lambda}} . \quad (62)$$

The geodesic equation is then in coordinate-independent notation

$$\frac{d\mathbf{p}}{d\lambda} = 0 \quad (63)$$

or in component form

$$\frac{dp^\kappa}{d\lambda} + \Gamma_{\mu\nu}^\kappa p^\mu p^\nu = 0 \quad (64)$$

which works for massless as well as massive particles.

Another way of writing this is

$$\frac{Dp^\kappa}{D\lambda} = 0 \quad (65)$$

where $D/D\lambda$ is the covariant affine derivative

$$\frac{D}{D\lambda} \equiv p^\nu D_\nu . \quad (66)$$

2.25 Affine distance

The affine parameter is also called the **affine distance**, because it provides a measure of distance along null geodesics. When you look at a scene with your eyes, you are looking along null geodesics, and the natural measure of distance to objects that you see is the affine distance. The freedom in the overall scaling of the affine distance is fixed by setting it equal to the proper distance near the observer in the observer's locally inertial rest frame.

In special relativity, the affine distance coincides with the perceived (e.g. binocular) distance to objects.

2.26 Riemann curvature tensor

The **Riemann curvature tensor** $R_{\kappa\lambda\mu\nu}$ is defined by the commutator of the covariant derivative acting on a contravariant 4-vector

$$\boxed{[D_\kappa, D_\lambda] A_\mu = R_{\kappa\lambda\mu\nu} A^\nu} \quad \text{is a tensor .} \quad (67)$$

The expression (40) for the covariant derivative yields the following formula for the Riemann tensor in terms of connection coefficients

$$\boxed{R_{\kappa\lambda\mu\nu} = \frac{\partial\Gamma_{\mu\nu\lambda}}{\partial x^\kappa} - \frac{\partial\Gamma_{\mu\nu\kappa}}{\partial x^\lambda} + \Gamma_{\mu\lambda}^\alpha \Gamma_{\alpha\nu\kappa} - \Gamma_{\mu\kappa}^\alpha \Gamma_{\alpha\nu\lambda}} \quad \text{is a tensor .} \quad (68)$$

This is the formula that allows the Riemann tensor to be calculated from the connection coefficients.

In flat (Minkowski) space, covariant derivatives reduce to partial derivatives, $D_\kappa \rightarrow \partial/\partial x^\kappa$, and

$$[D_\kappa, D_\lambda] \rightarrow \left[\frac{\partial}{\partial x^\kappa}, \frac{\partial}{\partial x^\lambda} \right] = 0 \quad \text{in flat space} \quad (69)$$

so that $R_{\kappa\lambda\mu\nu} = 0$ in flat space.

Comment: In quantum field theories (QED, QCD), the commutator of the gauge-covariant derivative is taken to be the field. In conventional GR, by contrast, the metric is taken to be the fundamental field, rather than the curvature. Another difference between quantum field theories and GR is that the Lagrangian of quantum field theories is taken to be quadratic in the field, whereas the Lagrangian of GR is taken to be linear in the curvature (specifically, the GR Lagrangian is the Ricci scalar R).

2.27 Symmetries of the Riemann tensor

In a locally inertial frame, the connection coefficients all vanish, $\Gamma_{\lambda\mu\nu} = 0$, but their partial derivatives, which are proportional to second derivatives of the metric tensor, do not vanish. Thus in a locally inertial frame the Riemann tensor is

$$\begin{aligned} R_{\kappa\lambda\mu\nu} &= \frac{\partial\Gamma_{\mu\nu\lambda}}{\partial x^\kappa} - \frac{\partial\Gamma_{\mu\nu\kappa}}{\partial x^\lambda} \\ &= \frac{1}{2} \left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 g_{\mu\lambda}}{\partial x^\kappa \partial x^\nu} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} + \frac{\partial^2 g_{\nu\kappa}}{\partial x^\lambda \partial x^\mu} \right) \\ &= \frac{1}{2} \left(\frac{\partial^2 g_{\mu\lambda}}{\partial x^\kappa \partial x^\nu} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} + \frac{\partial^2 g_{\nu\kappa}}{\partial x^\lambda \partial x^\mu} \right). \end{aligned} \quad (70)$$

You can check that the bottom line of equation (70):

1. is antisymmetric in $\kappa \leftrightarrow \lambda$,
2. is antisymmetric in $\mu \leftrightarrow \nu$,
3. is symmetric in $\kappa\lambda \leftrightarrow \mu\nu$,

4. has the property that the sum of the cyclic permutations of the last three indices vanishes

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\nu\lambda\mu} + R_{\kappa\mu\nu\lambda} = 0 . \quad (71)$$

The first three of these four symmetries can be summarized by the shorthand notation

$$R_{([\kappa\lambda][\mu\nu])} \quad (72)$$

in which $[]$ denotes anti-symmetrization and $()$ symmetrization. These symmetries imply that the Riemann tensor is a symmetric matrix of antisymmetric matrices. An antisymmetric matrix has 6 degrees of freedom. A symmetric matrix of these things is a 6×6 symmetric matrix, which has 21 degrees of freedom. The final, cyclic symmetry of the Riemann tensor, equation (71), removes 1 degree of freedom. Thus the Riemann tensor has a net 20 degrees of freedom.

Although the above symmetries were derived in a locally inertial frame, the fact that the Riemann tensor is a tensor means that the symmetries hold in any frame. If you prefer, you can add back the products of connection coefficients in equation (68), and check that the claimed symmetries remain.

2.28 Ricci tensor, Ricci scalar

The Ricci tensor $R_{\kappa\mu}$ and Ricci scalar R are the essentially unique contractions of the Riemann curvature tensor. The **Ricci tensor**, the compressive part of the Riemann tensor, is

$$\boxed{R_{\kappa\mu} \equiv g^{\lambda\nu} R_{\kappa\lambda\mu\nu}} \quad \text{is a tensor .} \quad (73)$$

The symmetries of the Riemann tensor imply that the Ricci tensor is symmetric

$$R_{\kappa\mu} = R_{\mu\kappa} \quad (74)$$

and therefore has 10 independent components.

The **Ricci scalar** is

$$\boxed{R \equiv g^{\kappa\mu} R_{\kappa\mu}} \quad \text{is a tensor (a scalar) .} \quad (75)$$

2.29 Einstein tensor

The **Einstein tensor** $G_{\kappa\mu}$ is defined by

$$\boxed{G_{\kappa\mu} \equiv R_{\kappa\mu} - \frac{1}{2} R g_{\kappa\mu}} \quad \text{is a tensor .} \quad (76)$$

The symmetry of the Ricci and metric tensors imply that the Einstein tensor is likewise symmetric

$$G_{\kappa\mu} = G_{\mu\kappa} . \quad (77)$$

The Einstein tensor has 10 independent components.

2.30 Bianchi identities

The Jacobi identity

$$[D_\kappa, [D_\lambda, D_\mu]] + [D_\lambda, [D_\mu, D_\kappa]] + [D_\mu, [D_\kappa, D_\lambda]] = 0 \quad (78)$$

implies the **Bianchi identities**

$$\boxed{D_\kappa R_{\lambda\mu\nu\pi} + D_\lambda R_{\mu\kappa\nu\pi} + D_\mu R_{\kappa\lambda\nu\pi} = 0} . \quad (79)$$

The Bianchi identities constitute a set of differential relations between the components of the Riemann tensor, which are distinct from the algebraic symmetries of the Riemann tensor.

2.31 Covariant conservation of the Einstein tensor

The most important consequence of the Bianchi identities is obtained from the following contraction

$$g^{\kappa\nu} g^{\lambda\pi} (D_\kappa R_{\lambda\mu\nu\pi} + D_\lambda R_{\mu\kappa\nu\pi} + D_\mu R_{\kappa\lambda\nu\pi}) = -D^\kappa R_{\kappa\mu} - D^\lambda R_{\lambda\mu} + D_\mu R = 0 \quad (80)$$

which implies that

$$D^\kappa G_{\kappa\mu} = 0 . \quad (81)$$

This equation is a primary motivation for the form of the Einstein equations, since it implies energy-momentum conservation.

2.32 Einstein equations

Einstein's equations are

$$\boxed{G_{\kappa\mu} = 8\pi G T_{\kappa\mu}} \quad \text{is a tensor equation} . \quad (82)$$

What motivates the form of Einstein's equations?

1. The equation is generally covariant;
2. The Bianchi identities guarantee conservation of energy-momentum;
3. The Einstein tensor depends on the lowest (second) order derivatives of the metric tensor that do not vanish in a locally inertial frame.

The covariant conservation of the Einstein tensor, equation (81), implies the conservation of energy-momentum

$$\boxed{D^k T_{\kappa\mu} = 0} . \quad (83)$$

By themselves, Einstein's equations do not constitute a closed set of equations: they specify only the relation between curvature and energy-momentum, generalizing Poisson's equation of Newtonian gravity. In general, other equations, such as Maxwell's equations of electromagnetism, and equations describing the microphysics of the energy-momentum, must be adjoined to form a closed set. Moreover, Einstein's equations do not describe gravitational waves, which are described by the Weyl tensor.

2.33 Summary of the path from metric to the energy-momentum tensor

1. Start by defining the metric $g_{\mu\nu}$.
2. Compute the connection coefficients $\Gamma_{\lambda\mu\nu}$ from equation (50).
3. Compute the Riemann tensor $R_{\kappa\lambda\mu\nu}$ from equation (68).
4. Compute the Ricci tensor $R_{\kappa\mu}$ from equation (73), the Ricci scalar R from equation (75), and the Einstein tensor $G_{\kappa\mu}$ from equation (76).
5. The Einstein equations (82) then imply the energy-momentum tensor $T_{\kappa\mu}$.

The path from metric to energy-momentum tensor is straightforward to program on a computer, but the results are typically messy and complicated, even for fairly simple spacetimes. Inverting the path to recover the metric from a given energy-momentum content is typically highly non-trivial, the subject of a huge literature.

2.34 Weyl tensor

The trace-free, or tidal, part of the Riemann curvature tensor defines the **Weyl tensor** $C_{\kappa\lambda\mu\nu}$

$$\boxed{C_{\kappa\lambda\mu\nu} \equiv R_{\kappa\lambda\mu\nu} - \frac{1}{2}(g_{\kappa\mu}R_{\lambda\nu} - g_{\kappa\nu}R_{\lambda\mu} + g_{\lambda\nu}R_{\kappa\mu} - g_{\lambda\mu}R_{\kappa\nu}) + \frac{1}{6}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu})} \text{ is a tensor.} \quad (84)$$

The Weyl tensor has 10 independent components. These 10 components together with the 10 components of the Ricci tensor account for the 20 components of the Riemann tensor. The Weyl tensor inherits the symmetries (72) of the Riemann tensor

$$C_{([\kappa\lambda][\mu\nu])} \cdot \quad (85)$$

Whereas the Einstein tensor necessarily vanishes in a region of spacetime where there is no energy-momentum, the Weyl tensor does not. The Weyl tensor expresses the presence of tidal gravitational forces, and of gravitational waves.

2.35 Evolution equations for the Weyl tensor

The Bianchi identities yield evolution equations for the Weyl tensor

$$D^\kappa C_{\kappa\lambda\mu\nu} = J_{\lambda\mu\nu} \quad (86)$$

with Weyl current

$$J_{\lambda\mu\nu} \equiv \frac{1}{2}(D_\mu G_{\lambda\nu} - D_\nu G_{\lambda\mu}) - \frac{1}{6}(g_{\lambda\nu}D_\mu G - g_{\lambda\mu}D_\nu G) \cdot \quad (87)$$

The evolution equations (86) can be regarded as the gravitational analog of Maxwell's equations of electromagnetism. The Weyl current $J_{\lambda\mu\nu}$ satisfies the conservation law

$$D^\lambda J_{\lambda\mu\nu} = 0 \quad (88)$$

in view of equation (86) and the antisymmetry of $C_{\kappa\lambda\mu\nu}$ with respect to the indices $\kappa\lambda$.

2.36 Energy-momentum tensor of an ideal fluid

The simplest non-trivial energy-momentum tensor is that of an **ideal fluid**. In this case $T_{\mu\nu}$ is isotropic in the locally inertial rest frame of the fluid, taking the form

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad (89)$$

where

$$\begin{aligned} \rho & \text{ is the proper mass-energy density ,} \\ p & \text{ is the proper pressure .} \end{aligned} \quad (90)$$

The expression (89) is valid only in the locally inertial rest frame of the fluid. An expression that is valid in any frame is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \quad (91)$$

where u^μ is the 4-velocity of the fluid. Equation (91) is valid because it is a tensor equation, and it is true in the locally inertial rest frame, where $u^\mu = \{1, 0, 0, 0\}$.

2.37 Newtonian limit

The Newtonian limit is obtained in the limit of a weak gravitational field and non-relativistic (pressureless) matter. In Cartesian coordinates, the metric in the Newtonian limit is

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \quad (92)$$

in which

$$\Phi(x, y, z) = \text{Newtonian potential} \quad (93)$$

is a function only of the spatial coordinates x, y, z , not of time t .

For this metric, to first order in the potential Φ the only non-vanishing component of the Einstein tensor is the time-time component

$$G_{tt} = 2\nabla^2\Phi \quad (94)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the usual 3D Laplacian operator. This component (94) of the Einstein tensor plugged into Einstein's equations (82) implies Poisson's equation (4).

2.38 Geodesic deviation

This subsection on geodesic deviation is included in these notes not because the equation of geodesic deviation is crucial to everyday calculations in GR, but rather for two reasons. First, the equation offers insight into the physical meaning of the Riemann tensor. Second, the derivation of the equation offers a fine illustration of the fact that in GR, whenever you

take differences at infinitesimally separated points in space or time, you should always take covariant differences.

Consider two objects that are free-falling along two infinitesimally separated geodesics. In flat space the acceleration between the two objects would be zero, but in curved space the curvature induces a finite acceleration between the two objects. This is how an observer can measure curvature, at least in principle: set up an ensemble of objects initially at rest a small distance away from the observer in the observer's locally inertial frame, and watch how the objects begin to move. The equation (100) that describes this acceleration between objects an infinitesimal distance apart is called the **equation of geodesic deviation**.

The covariant difference in the velocities of two objects an infinitesimal distance δx^μ apart is

$$\frac{D\delta x^\mu}{D\tau} = \delta u^\mu . \quad (95)$$

In GR, the ordinary difference between vectors at two points a small interval apart is not a physically meaningful thing, because the frames of reference at the two points are different. The only physically meaningful difference is the covariant difference, which is the difference in the two vectors parallel-transported across the gap between them. It is only this covariant difference that is independent of the frame of reference. On the left hand side of equation (95), the proper time derivative must be the covariant proper time derivative, $D/D\tau = u^\lambda D_\lambda$. On the right hand side of equation (95), the difference in the 4-velocity at two points δx^κ apart must be the covariance difference $\delta = \delta x^\kappa D_\kappa$. Thus equation (95) means explicitly the covariant equation

$$u^\lambda D_\lambda \delta x^\mu = \delta x^\kappa D_\kappa u^\mu . \quad (96)$$

To derive the equation of geodesic deviation, first vary the geodesic equation (I've put the index μ downstairs in the following equations, so that the final equation (100) looks cosmetically better, but of course since everything is covariant the μ index could just as well be put upstairs everywhere):

$$\begin{aligned} 0 &= \delta \frac{Du_\mu}{D\tau} \\ &= \delta x^\kappa D_\kappa u^\lambda D_\lambda u_\mu \\ &= \delta u^\lambda D_\lambda u_\mu + \delta x^\kappa u^\lambda D_\kappa D_\lambda u_\mu . \end{aligned} \quad (97)$$

On the second line, the covariant difference δ between quantities a small distance δx^κ apart has been set equal to $\delta x^\kappa D_\kappa$, while $D/D\tau$ has been set equal to the covariant time derivative $u^\lambda D_\lambda$ along the geodesic. On the last line, $\delta x^\kappa D_\kappa u^\lambda$ has been replaced by δu^μ . Next, consider the covariant acceleration of the interval δx_μ , which is the covariant proper time derivative of the covariant velocity difference δu_μ :

$$\begin{aligned} \frac{D^2 \delta x_\mu}{D\tau^2} &= \frac{D\delta u_\mu}{D\tau} \\ &= u^\lambda D_\lambda \delta x^\kappa D_\kappa u_\mu \\ &= \delta u^\kappa D_\kappa u_\mu + \delta x^\kappa u^\lambda D_\lambda D_\kappa u_\mu . \end{aligned} \quad (98)$$

As in the previous equation (97), on the second line $D/D\tau$ has been set equal to $u^\lambda D_\lambda$, while δ has been set equal to $\delta x^\kappa D_\kappa$. On the last line, $u^\lambda D_\lambda \delta x^\kappa$ has been set equal to δu^μ , equation (96). Subtracting (97) from (98) gives

$$\frac{D^2 \delta x_\mu}{D\tau^2} = \delta x^\kappa u^\lambda [D_\lambda, D_\kappa] u_\mu \quad (99)$$

or equivalently

$$\frac{D^2 \delta x_\mu}{D\tau^2} + R_{\kappa\lambda\mu\nu} \delta x^\kappa u^\lambda u^\nu = 0 \quad (100)$$

which is the desired equation of geodesic deviation.

2.39 Commutator of the covariant derivative revisited

The commutator of the covariant derivative is of fundamental importance because it defines what is meant by the field in gauge theories. It was seen above that the commutator of the covariant derivative acting on a scalar defined the torsion tensor, equation (45), which GR assumes vanishes, while the commutator of the covariant derivative acting on a vector defined the Riemann tensor, equation (67). Does the commutator of the covariant derivative acting on a general tensor introduce any other distinct tensor? No: the torsion and Riemann tensors completely define the action of the commutator of the covariant derivative on any tensor, equation (104).

The general expression (104) for the commutator of the covariant derivative reveals the meaning of the torsion and Riemann tensors. The torsion and Riemann tensors describe respectively the displacement and the Lorentz transformation experienced by an object when parallel-transported around a curve. Displacement and Lorentz transformations together constitute the Poincaré group, the complete group of symmetries of flat spacetime.

How can an object detect a displacement when parallel-transported around a curve? If you go around a curve back to the same coordinate in spacetime where you began, won't you necessarily be at the same position? This is a question that goes to heart of the meaning of spacetime. To answer the question, you have to consider how fundamental particles are able to detect position, orientation, and velocity. Classically, particles may be structureless points, but quantum mechanically, particles possess frequency, wavelength, spin, and (in the relativistic theory) boost, and presumably it is these properties that allow particles to "measure" the properties of the spacetime in which they live. Specifically, a Dirac spinor (relativistic spin- $\frac{1}{2}$ particle) has 8 degrees of freedom, of which 6 define a Lorentz transformation (a Lorentz rotor, a member of the group of spin- $\frac{1}{2}$ Lorentz transformations), and the remaining 2 comprise a complex number $\sim e^{-i\mathbf{p}\cdot\mathbf{x}}$ whose phase encodes the displacement.

In the presence of torsion, the expression for the connection coefficients $\Gamma_{\lambda\mu\nu}$ is, from equation (50),

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + S_{\lambda\mu\nu} + S_{\mu\nu\lambda} + S_{\nu\mu\lambda} \right). \quad (101)$$

The first part $\frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$ of this expression is called the Christoffel symbol of the first kind [the same thing with the first index raised, $\frac{1}{2}g^{\kappa\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda})$, is called the Christoffel symbol of the second kind], while the second part $\frac{1}{2}(S_{\lambda\mu\nu} + S_{\mu\nu\lambda} + S_{\nu\mu\lambda})$ is called the contortion (not contorsion!) tensor.

If torsion does not vanish, then the commutator of the covariant derivative acting on a contravariant 4-vector is

$$\boxed{[D_\kappa, D_\lambda] A_\mu = S_{\kappa\lambda}^\nu D_\nu A_\mu + R_{\kappa\lambda\mu\nu} A^\nu} \quad \text{is a tensor} \quad (102)$$

where the Riemann tensor $R_{\kappa\lambda\mu\nu}$ is given in terms of the connection coefficients by the same formula (68) as before, but the connection coefficients $\Gamma_{\lambda\mu\nu}$ themselves are given by (101). The Riemann tensor is still antisymmetric in each of $\kappa \leftrightarrow \lambda$ and $\mu \leftrightarrow \nu$, but with torsion it is no longer symmetric in $\kappa\lambda \leftrightarrow \mu\nu$. In other words, the symmetries of the Riemann tensor with torsion are

$$R_{[\kappa\lambda][\mu\nu]} \cdot \quad (103)$$

As a matrix of antisymmetric tensors, the Riemann tensor with torsion has $6 \times 6 = 36$ degrees of freedom. Because the Riemann tensor $R_{\kappa\lambda\mu\nu}$ is no longer symmetric in $\kappa\lambda \leftrightarrow \mu\nu$, the Ricci tensor $R_{\kappa\mu} \equiv g^{\lambda\nu} R_{\kappa\lambda\mu\nu}$ is no longer symmetric, and likewise the Einstein tensor $G_{\kappa\mu} \equiv R_{\kappa\mu} - \frac{1}{2}Rg_{\kappa\mu}$ is no longer symmetric. Evidently the antisymmetric part of the Einstein tensor depends on torsion.

Acting on a general tensor, the commutator of the covariant derivative is

$$\boxed{[D_\kappa, D_\lambda] A_{\mu\nu\dots}^{\pi\rho\dots} = S_{\kappa\lambda}^\alpha D_\alpha A_{\mu\nu\dots}^{\pi\rho\dots} + R_{\kappa\lambda\mu}{}^\alpha A_{\alpha\nu\dots}^{\pi\rho\dots} + R_{\kappa\lambda\nu}{}^\alpha A_{\mu\alpha\dots}^{\pi\rho\dots} - R_{\kappa\lambda\alpha}{}^\pi A_{\mu\nu\dots}^{\alpha\rho\dots} - R_{\kappa\lambda\alpha}{}^\rho A_{\mu\nu\dots}^{\pi\alpha\dots}} \cdot \quad (104)$$

In more abstract notation, the commutator of the covariant derivative is the operator

$$[D_\kappa, D_\lambda] = \mathbf{S}_{\kappa\lambda} \cdot \mathbf{D} + \mathcal{R}_{\kappa\lambda} \quad (105)$$

where $\mathbf{S}_{\kappa\lambda} \equiv g_\mu S_{\kappa\lambda}^\mu$ and $\mathbf{D} \equiv g^\mu D_\mu$, and the Riemann curvature operator $\mathcal{R}_{\kappa\lambda}$ is an operator whose action on any tensor is specified by equation (104). The action of the operator $\mathcal{R}_{\kappa\lambda}$ is similar to that of the covariant derivative (41): there's a positive R term for each contravariant index, and a negative R term for each covariant index. The action of $\mathcal{R}_{\kappa\lambda}$ on a scalar is zero, which reflects the fact that a scalar is unchanged by a Lorentz transformation.