6 General Relativistic Perturbation Theory

6.1 Concept Questions

1. Why do general relativistic perturbation theory using the tetrad formalism as opposed to the coordinate approach?

2. Why is the tetrad metric γ_{mn} assumed fixed in the presence of perturbations?

3. Are the tetrad axes $\boldsymbol{\gamma}_m$ fixed under a perturbation?

4. Is it true that the tetrad components φ_{mn} of a perturbation are (anti-)symmetric in $m \leftrightarrow n$ if and only if its coordinate components $\varphi_{\mu\nu}$ are (anti-)symmetric in $\mu \leftrightarrow \nu$?

5. Does an unperturbed quantity, such as the unperturbed metric $\overset{0}{g}_{\mu\nu}$, change under an infinitesimal coordinate gauge transformation?

6. How can the vierbein perturbation φ_{mn} be considered a tetrad tensor field if it changes under an infinitesimal coordinate gauge transformations?

7. What properties of the unperturbed spacetime allow decomposition of perturbations into independently evolving Fourier modes?

8. What properties of the unperturbed spacetime allow decomposition of perturbations into independently evolving scalar, vector, and tensor modes?

9. In what sense do scalar, vector, and tensor modes have spin 0, 1, and 2 respectively?

10. Tensor modes represent gravitational waves that propagate at the speed of light. If scalar and vector modes also propagate at the speed of light, do these not also constitute gravitational waves?

11. Equation (81) defines the mass M of a body as what a distant observer would measure from its gravitational potential. Similarly equation (89) defines the angular momentum $\boldsymbol{L}$ of a body as what a distant observer would measure from the dragging of inertial frames. In what sense are these definitions legitimate?

12. Can an observer far from a body detect the difference between the scalar potentials Ψ and Φ produced by the body?

13. If a gravitational wave is a wave of spacetime itself, distorting the very rulers and clocks that measure spacetime, how is it possible to measure gravitational waves at all?

14. Have gravitational waves been detected?

15. If gravitational waves carry energy-momentum, then can gravitational waves be present in a region of spacetime with vanishing energy-momentum tensor, $T_{mn} = 0$?

16. Why do the wavelengths of perturbations in cosmology expand with the Universe, whereas perturbations in Minkowski space do not expand?

17. What does power spectrum mean?

18. Why is the power spectrum a good way to characterize the amplitude of fluctuations?

19. Why is the power spectrum of fluctuations of the Cosmic Microwave Background (CMB) plotted as a function of harmonic number?

20. What causes the acoustic peaks in the power spectrum of fluctuations of the CMB?

21. Are there acoustic peaks in the power spectrum of matter (galaxies) today?

22. What sets the scale of the first peak in the power spectrum of the CMB? [What sets the physical scale? Then what sets the angular scale?]

23. The odd peaks (including the first peak) in the CMB power spectrum are compression peaks, while the even peaks are rarefaction peaks. Why does a rarefaction produce a peak, not a trough?

24. Why is the first peak the most prominent? Why do higher peaks generally get progressively weaker?

25. The third peak is about as strong as the second peak? Why?

26. The matter power spectrum reaches a maximum at a scale that is slightly larger than the scale of the first baryonic acoustic peak. Why?

27. The physical density of species x at the time of recombination is proportional to $\Omega_x h^2$ where Ω_x is the ratio of the actual to critical density of species x at the present time, and $h \equiv H_0/100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ is the present day Hubble constant. Explain.

28. How does changing the baryon density $\Omega_b h^2$ affect the CMB power spectrum?

29. How does changing the non-baryonic cold dark matter density $\Omega_c h^2$, without changing the baryon density $\Omega_b h^2$, affect the CMB power spectrum?

30. What effects do neutrinos have on perturbations?

31. How does changing the curvature Ω_k affect the CMB power spectrum?

32. How does changing the dark energy Ω_Λ affect the CMB power spectrum?

6.2 What's important?

This section of the notes describes the elements of perturbation theory in GR, using the tetrad formalism. It covers gravitational waves, and cosmological perturbation theory.

1. Getting your brain around coordinate and tetrad gauge transformations.

2. A central aim of general relativistic perturbation theory is to identify the coordinate and tetrad gauge-invariant perturbations, since only these have physical meaning.

3. A second central aim is to classify perturbations into independently evolving modes, to the extent that this is possible.

4. In background spacetimes with spatial translation and rotation symmetry, which includes Minkowski space and the Friedmann-Roberston-Walker metric of cosmology, modes decompose into independently evolving scalar (spin-0), vector (spin-1), and tensor (spin-2) modes. In background spacetimes without spatial translation and rotation symmetry, such as black holes, scalar, vector, and tensor modes scatter off the curvature of space, and therefore mix with each other.

5. In background spacetimes with spatial translation and rotation symmetry, there are 6 algebraic combinations of metric coefficients that are coordinate and tetrad gauge-invariant, and therefore represent physical perturbations. There are 2 scalar modes, 2 vector modes, and 2 tensor modes. A spin-m mode varies as $e^{im\chi}$ where χ is the rotational angle about the spatial wavevector $\boldsymbol{k}$ of the mode.

6. In background spacetimes without spatial translation and rotation symmetry, the coordinate and tetrad gauge-invariant perturbations are not algebraic combinations of the metric coefficients, but rather combinations that involve first and second derivatives of the metric coefficients. Gravitational waves are described by the Weyl tensor, which can be decomposed into 5 complex components, with spin 0, ± 1, and ± 2. The spin-± 2 components describe propagating gravitational waves, while the spin-0 and spin-± 1 components describe the non-propagating gravitational field near a source.

7. The preeminent application of general relativistic perturbation theory is to cosmology. Coupled with physics that is either well understood (such as photon-electron scattering) or straightforward to model even without a deep understanding (such as the dynamical behavior of non-baryonic dark matter and dark energy), the theory has yielded predictions that are in spectacular agreement with observations of fluctuations in the CMB and in the large scale distribution of galaxies.

6.3 Fundamentals

This section sets out the fundamental formulae of general relativistic perturbation theory in all generality. The purpose is to investigate what happens when a given background spacetime is perturbed by a small amount. The perturbation is supposed to be small, in the sense that quantities quadratic in the perturbation can be neglected.

6.3.1 Notation for perturbations

A $_0$ (zero) overscript signifies an unperturbed quantity, while a $_1$ (one) overscript signifies a perturbation. No overscript means the full quantity, including both unperturbed and perturbed parts. An overscript is attached only where necessary. Thus if the unperturbed part of a quantity is zero, then no overscript is needed, and none is attached.

6.3.2 Vierbein perturbation

Let the vierbein perturbation φ_{mn} be defined so that the perturbed vierbein is

$$\boxed{e_m{}^\mu = (\delta_m^n - \varphi_m{}^n)\overset{0}{e}_n{}^\mu} \ , \tag{1}$$

with corresponding inverse

$$\boxed{e^m{}_\mu = (\delta_n^m + \varphi_n{}^m)\overset{0}{e}{}^n{}_\mu} \ . \tag{2}$$

Since the perturbation $\varphi^m{}_\mu$ is already of linear order, to linear order its indices can be raised and lowered with the unperturbed metric, and transformed between tetrad and coordinate frames with the unperturbed vierbein. In practice it proves convenient to work with the covariant tetrad-frame components φ_{mn} of the vierbein pertubation

$$\varphi_{mn} = \varphi_m{}^\mu \, \overset{0}{e}_{n\mu} \ . \tag{3}$$

The perturbation φ_{mn} can be regarded as a tetrad tensor field defined on the unperturbed background.

6.3.3 Gauge transformations

The vierbein perturbation φ_{mn} has 16 degrees of freedom, but only 6 of these degrees of freedom correspond to real physical perturbations, since 6 degrees of freedom are associated with arbitrary infinitesimal changes in the choice of tetrad, which is to say arbitrary infinitesimal Lorentz transformations, and a further 4 degrees of freedom are associated with arbitrary infinitesimal changes in the coordinates.

In the context of perturbation theory, these infinitesimal tetrad and coordinate transformations are called **gauge transformations**. Real physical perturbations are perturbations that are **gauge-invariant** under both tetrad and coordinate gauge transformations.

6.3.4 Tetrad metric assumed constant

In the tetrad formalism, tetrad axes $\boldsymbol{\gamma}_m$ are introduced as locally inertial (or other physically motivated) axes attached to an observer. The axes enable quantities to be projected into the frame of the observer. In a spacetime buffeted by perturbations, it is natural for an observer to cling to the rock provided by the locally inertial (or other) axes, as opposed to allowing the axes to bend with the wind. For example, when a gravitational wave goes by, the tidal compression and rarefaction causes the proper distance between two freely falling test masses to oscillate. It is natural to choose the tetrad so that it continues to measure proper times and distances in the perturbed spacetime.

In these notes on general relativistic perturbation theory, the tetrad metric will be taken to be constant everywhere, and unchanged by a perturbation

$$\boxed{\gamma_{mn} = \overset{0}{\gamma}_{mn} = \text{constant}} \ . \tag{4}$$

For example, if the tetrad is orthonormal, then the tetrad metric is constant, the Minkowski metric η_{mn}. However, the tetrad could also be some other tetrad for which the tetrad metric is constant, such as a spinor tetrad (§6.6.1), or a Newman-Penrose tetrad (§6.7.1).

It should be remarked that equation (4) does not cover all possibilities. In the ADM formalism, for example, the tetrad frame is tied to the coordinate system, and equation (4) is not true.

6.3.5 Perturbed coordinate metric

The perturbed coordinate metric is

$$\begin{aligned} g_{\mu\nu} &= \gamma_{mn} \, e^m{}_\mu e^n{}_\nu \\ &= \gamma_{kl} (\delta^k_m + \varphi_m{}^k) \overset{0}{e}{}^m{}_\mu (\delta^l_n + \varphi_n{}^l) \overset{0}{e}{}^n{}_\nu \\ &= \overset{0}{g}_{\mu\nu} + \varphi_{\mu\nu} + \varphi_{\nu\mu} \ . \end{aligned} \tag{5}$$

Thus the perturbation of the coordinate metric depends only on the symmetric part of the vierbein perturbation φ_{mn}, not the antisymmetric part

$$\boxed{\overset{1}{g}_{\mu\nu} = \varphi_{\mu\nu} + \varphi_{\nu\mu}} \ . \tag{6}$$

6.3.6 Tetrad gauge transformations

Under an infinitesimal tetrad transformation, the covariant vierbein perturbations φ_{mn} transform as

$$\varphi_{mn} \to \varphi_{mn} + \epsilon_{mn} \ , \tag{7}$$

where ϵ_{mn} is the generator of a Lorentz transformation, which is to say an arbitrary antisymmetric tensor. Thus the antisymmetric part $\varphi_{mn} - \varphi_{nm}$ of the covariant perturbation φ_{mn} is arbitrarily adjustable through an infinitesimal tetrad transformation, while the symmetric part $\varphi_{mn} + \varphi_{nm}$ is tetrad gauge-invariant.

It is easy to see when a quantity is tetrad gauge-invariant: it is tetrad gauge-invariant if and only if it depends only on the symmetric part of the vierbein perturbation, not on the antisymmetric part. Evidently the perturbation (6) to the coordinate metric $g_{\mu\nu}$ is tetrad gauge-invariant. This is as it should be, since the coordinate metric $g_{\mu\nu}$ is a coordinate-frame quantity, independent of the choice of tetrad frame.

If only tetrad gauge-invariant perturbations are physical, why not just discard tetrad perturbations (the antisymmetric part of φ_{mn}) altogether, and work only with the tetrad gauge-invariant part (the symmetric part of φ_{mn})? The answer is that tetrad-frame quantities such as the tetrad-frame Einstein tensor do change under tetrad gauge transformations (infinitesimal Lorentz transformations of the tetrad). It is true that the only physical perturbations of the Einstein tensor are those combinations of it that are tetrad gauge-invariant. But in order to identify these tetrad gauge-invariant combinations, it is necessary to carry through the dependence on the non-tetrad-gauge-invariant part, the antisymmetric part of φ_{mn}.

Much of the professional literature on general relativistic perturbation theory works with the traditional coordinate formalism, as opposed to the tetrad formalism. The term "gauge-invariant" then means coordinate gauge-invariant, as opposed to both coordinate and tetrad gauge-invariant. This is fine as far as it goes: the coordinate approach is perfectly able to identify physical perturbations versus gauge perturbations. However, there still remains the problem of projecting the perturbations into the frame of an observer, so ultimately the issue of perturbations of the observer's frame, tetrad perturbations, must be faced.

6.3.7 Coordinate gauge transformations

A coordinate gauge transformation is a transformation of the coordinates x^μ by an infinitesimal shift ϵ^μ

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu \ . \tag{8}$$

You should not think of this as shifting the underlying spacetime around; rather, it is just a change of the coordinate system, which leaves the underlying spacetime unchanged. Because the shift ϵ^μ is, like the vierbein perturbations φ_{mn}, already of linear order, its indices can be raised and lowered with the unperturbed metric, and transformed between coordinate and tetrad frames with the unperturbed vierbein. Thus the shift ϵ^μ can be regarded as a vector field defined on the unperturbed background. The tetrad components ϵ^m of the shift ϵ^μ are

$$\epsilon^m = \overset{0}{e}{}^m{}_\mu \, \epsilon^\mu \ . \tag{9}$$

Physically, the tetrad-frame shift ϵ^m is the shift measured in locally inertial coordinates

$$\xi^m \rightarrow \xi'^m = \xi^m + \epsilon^m \ . \tag{10}$$

6.3.8 Coordinate gauge transformation of a coordinate vector

Under a coordinate transformation (8), a coordinate-frame 4-vector $A^\mu(x)$ transforms in the usual way as

$$A^\mu(x) \to A'^\mu(x') = A^\kappa(x) \frac{\partial x'^\mu}{\partial x^\kappa} = A^\mu(x) + A^\kappa(x) \frac{\partial \epsilon^\mu}{\partial x^\kappa} \ . \tag{11}$$

Here the tensor $A'^\mu(x')$ is evaluated at position x', which is the same as the original physical position x since all that has changed is the coordinates, not the physical position. However, in perturbation theory, quantities are evaluated at coordinate position x, not x'. The value of A'^μ at x is related to that at x' by

$$A'^\mu(x) = A'^\mu(x' - \epsilon) = A'^\mu(x') - \epsilon^\kappa \frac{\partial A'^\mu}{\partial x^\kappa} \ . \tag{12}$$

Since ϵ^κ is a small quantity, and A'^μ differs from A^μ by a small quantity, the last term $\epsilon^\kappa \, \partial A'^\mu / \partial x^\kappa$ in equation (12) can be replaced by $\epsilon^\kappa \, \partial A^\mu / \partial x^\kappa$ to linear order. Putting equations (11) and (12) together shows that the 4-vector A^μ changes under a coordinate gauge transformation (8) as

$$\boxed{A^\mu(x) \to A'^\mu(x) = A^\mu + A^\kappa \frac{\partial \epsilon^\mu}{\partial x^\kappa} - \epsilon^\kappa \frac{\partial A^\mu}{\partial x^\kappa}} \ . \tag{13}$$

6.3.9 Lie derivative

The change in the coordinate 4-vector A^μ on the right hand side of equation (13) is called the **Lie derivative** of A^μ along the direction ϵ^κ, and it is designated by the operator $\mathcal{L}_\epsilon$

$$\boxed{\mathcal{L}_\epsilon A^\mu \equiv A^\kappa \frac{\partial \epsilon^\mu}{\partial x^\kappa} - \epsilon^\kappa \frac{\partial A^\mu}{\partial x^\kappa}} \ . \tag{14}$$

The Lie derivative has the important property of being a tensor, which is one reason that it merits a special name. Translating from ordinary partial derivatives to covariant derivatives yields the following expression for the Lie derivative in covariant form

$$\boxed{\mathcal{L}_\epsilon A^\mu = A^\kappa D_\kappa \epsilon^\mu - \epsilon^\kappa D_\kappa A^\mu + A^\kappa \epsilon^\lambda S^\mu_{\kappa\lambda}} \quad \textit{is a tensor} \ , \tag{15}$$

where $S^\mu_{\kappa\lambda}$ is the torsion. If torsion vanishes, as GR assumes, then the Lie derivative of a 4-vector is

$$\mathcal{L}_\epsilon A^\mu = A^\kappa D_\kappa \epsilon^\mu - \epsilon^\kappa D_\kappa A^\mu \quad \textit{is a tensor} \ . \tag{16}$$

More generally, under a coordinate gauge transformation (8), a coordinate tensor $A^{\kappa\lambda...}_{\mu\nu...}$ transforms as

$$\boxed{A^{\kappa\lambda...}_{\mu\nu...}(x) \to A'^{\kappa\lambda...}_{\mu\nu...}(x) = A^{\kappa\lambda...}_{\mu\nu...} + \mathcal{L}_\epsilon A^{\kappa\lambda...}_{\mu\nu...}} \tag{17}$$

where the Lie derivative is defined by

$$\boxed{\mathcal{L}_\epsilon A^{\kappa\lambda...}_{\mu\nu...} \equiv A^{\alpha\lambda...}_{\mu\nu...} \frac{\partial \epsilon^\kappa}{\partial x^\alpha} + A^{\kappa\alpha...}_{\mu\nu...} \frac{\partial \epsilon^\lambda}{\partial x^\alpha} ... - A^{\kappa\lambda...}_{\alpha\nu...} \frac{\partial \epsilon^\alpha}{\partial x^\mu} - A^{\kappa\lambda...}_{\mu\alpha...} \frac{\partial \epsilon^\alpha}{\partial x^\nu} - \epsilon^\alpha \frac{\partial A^{\kappa\lambda...}_{\mu\nu...}}{\partial x^\alpha}} \ . \tag{18}$$

In covariant form, equation (18) is

$$\boxed{\begin{aligned}\mathcal{L}_\epsilon A^{\kappa\lambda\ldots}_{\mu\nu\ldots} = A^{\alpha\lambda\ldots}_{\mu\nu\ldots} D_\alpha \epsilon^\kappa + A^{\kappa\alpha\ldots}_{\mu\nu\ldots} D_\alpha \epsilon^\lambda \ldots - A^{\kappa\lambda\ldots}_{\alpha\nu\ldots} D_\mu \epsilon^\alpha - A^{\kappa\lambda\ldots}_{\mu\alpha\ldots} D_\nu \epsilon^\alpha \ldots - \epsilon^\alpha D_\alpha A^{\kappa\lambda\ldots}_{\mu\nu\ldots} \\ + \left(A^{\alpha\lambda\ldots}_{\mu\nu\ldots} S^\kappa_{\alpha\beta} + A^{\kappa\alpha\ldots}_{\mu\nu\ldots} S^\lambda_{\alpha\beta} \ldots - A^{\kappa\lambda\ldots}_{\alpha\nu\ldots} S^\alpha_{\mu\beta} - A^{\kappa\lambda\ldots}_{\mu\alpha\ldots} S^\alpha_{\nu\beta} \right) \epsilon^\beta\end{aligned}} \tag{19}$$

which *is* a tensor, and which in the case of vanishing torsion, as GR assumes, reduces to

$$\mathcal{L}_\epsilon A^{\kappa\lambda\ldots}_{\mu\nu\ldots} = A^{\alpha\lambda\ldots}_{\mu\nu\ldots} D_\alpha \epsilon^\kappa + A^{\kappa\alpha\ldots}_{\mu\nu\ldots} D_\alpha \epsilon^\lambda \ldots - A^{\kappa\lambda\ldots}_{\alpha\nu\ldots} D_\mu \epsilon^\alpha - A^{\kappa\lambda\ldots}_{\mu\alpha\ldots} D_\nu \epsilon^\alpha \ldots - \epsilon^\alpha D_\alpha A^{\kappa\lambda\ldots}_{\mu\nu\ldots} \ \textit{is}\text{ a tensor} \ . \tag{20}$$

As its name suggests, the Lie derivative acts like a derivative: it is linear, and it satisfies the Leibniz rule.

6.3.10 Coordinate gauge transformation of a tetrad vector

A tetrad-frame 4-vector A^m is a coordinate-invariant quantity, and therefore acts like a coordinate scalar under a coordinate gauge transformation (8)

$$\boxed{A^m(x) \to A'^m(x) = A^m - \epsilon^\alpha \frac{\partial A^m}{\partial x^\alpha} = A^m - \epsilon^a \partial_a A^m} \ . \tag{21}$$

The change $-\epsilon^a \partial_a A^m$ is a coordinate tensor (a coordinate scalar), but it is *not* a tetrad tensor. More generally, a tetrad-frame tensor $A^{kl\ldots}_{mn\ldots}$ transforms under a coordinate gauge transformation (8) as

$$\boxed{A^{kl\ldots}_{mn\ldots}(x) \to A'^{kl\ldots}_{mn\ldots}(x) = A^{kl\ldots}_{mn\ldots} - \epsilon^a \partial_a A^{kl\ldots}_{mn\ldots}} \ . \tag{22}$$

Again, the change $-\epsilon^a \partial_a A^{kl\ldots}_{mn\ldots}$ is a coordinate tensor (a coordinate scalar), but *not* a tetrad tensor.

6.3.11 Coordinate gauge transformation of the vierbein

The inverse vierbein $e^m{}_\mu$ equals the scalar product of the tetrad and coordinate axes, $e^m{}_\mu = \boldsymbol{\gamma}^m \cdot \boldsymbol{g}_\mu$. Therefore the transformation of the vierbein under a coordinate gauge transformation (8) follows from the transformations of $\boldsymbol{\gamma}^m$ and $\boldsymbol{g}_\mu$. The tetrad axes $\boldsymbol{\gamma}^m$ transform in accordance with (21) as

$$\begin{aligned}\boldsymbol{\gamma}^m \to \boldsymbol{\gamma}'^m &= \boldsymbol{\gamma}^m - \epsilon^k \partial_k \boldsymbol{\gamma}^m \\ &= \boldsymbol{\gamma}^m + \epsilon^k \Gamma^m_{nk} \boldsymbol{\gamma}^n \ .\end{aligned} \tag{23}$$

The coordinate axes $\boldsymbol{g}_\mu$ transform in accordance with (17) and (19) as

$$\begin{aligned}\boldsymbol{g}_\mu \to \boldsymbol{g}'_\mu &= \boldsymbol{g}_\mu + \mathcal{L}_\epsilon \boldsymbol{g}_\mu \\ &= \boldsymbol{g}_\mu - \boldsymbol{g}_\alpha D_\mu \epsilon^\alpha - \epsilon^\alpha D_\alpha \boldsymbol{g}_\mu - \boldsymbol{g}_\alpha S^\alpha_{\mu\kappa} \epsilon^\kappa \\ &= \boldsymbol{g}_\mu - \boldsymbol{g}_\alpha D_\mu \epsilon^\alpha - \boldsymbol{g}_\alpha S^\alpha_{\mu\kappa} \epsilon^\kappa \ ,\end{aligned} \tag{24}$$

where the term $\epsilon^\alpha D_\alpha \boldsymbol{g}_\mu$ in the expression on the second line vanishes because the axes $\boldsymbol{g}_\mu$ are by definition covariantly constant, $D_\alpha \boldsymbol{g}_\mu = 0$. It follows from (23) and (24) that the inverse vierbein $e^m{}_\mu$ transforms under an infinitesimal coordinate gauge transformation (8) as

$$\begin{aligned} e^m{}_\mu \rightarrow e'_m{}^\mu &= \boldsymbol{\gamma}'^m \cdot \boldsymbol{g}'_\mu \\ &= e^m{}_\mu + \left(- D_n \epsilon^m - S^m_{nk} \epsilon^k + \Gamma^m_{nk} \epsilon^k \right) \overset{0}{e}{}^n{}_\mu \ . \end{aligned} \tag{25}$$

From equation (25) and the definition (2) of the vierbein perturbations φ_{mn}, it follows that the vierbein perturbations transform under a coordinate gauge transformation (8) as

$$\boxed{\varphi_{mn} \rightarrow \varphi'_{mn} = \varphi_{mn} - \partial_m \epsilon_n + (\Gamma_{knm} + \Gamma_{nmk} - S_{nmk}) \, \epsilon^k} \ , \tag{26}$$

in which ϵ^k are the tetrad components of the coordinate shift, and Γ_{kmn} are tetrad connection coefficients.

6.3.12 Coordinate gauge transformation of the metric

The tetrad metric γ_{mn} transforms under an infinitesimal coordinate gauge transformation (8) as

$$\gamma_{mn} \rightarrow \gamma'_{mn} = \gamma_{mn} - (\Gamma_{mnk} + \Gamma_{nmk}) \epsilon^k = \gamma_{mn} \ , \tag{27}$$

where the last expression is true because the tetrad metric γ_{mn} is being assumed constant, equation (4), in which case $\Gamma_{mnk} + \Gamma_{nmk} = \partial_k \gamma_{nm} = 0$.

The coordinate metric $g_{\mu\nu}$ transforms under an infinitesimal coordinate gauge transformation (8) as

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + \mathcal{L}_\epsilon g_{\mu\nu} = g_{\mu\nu} - D_\mu \epsilon_\nu - D_\nu \epsilon_\mu - (S_{\mu\nu\kappa} + S_{\nu\mu\kappa}) \epsilon^\kappa \ . \tag{28}$$

6.4 Scalar, vector, tensor decomposition

In the particular case that the unperturbed spacetime is spatially homogeneous and isotropic, which includes not only Minkoswki space but also the important case of the cosmological Friedmann-Robertson-Walker metric, perturbations decompose into independently evolving scalar (spin-0), vector (spin-1), and tensor (spin-2) modes.

6.4.1 Decomposition of a vector in flat 3D space

Theorem: In flat 3-dimensional space, a 3-vector $\boldsymbol{w}$ can be decomposed into a sum of scalar and vector parts

$$\boxed{\boldsymbol{w} = \boldsymbol{\partial} w_\parallel + \boldsymbol{w}_\perp} \ . \tag{29}$$

In this context, the term **vector** signifies a 3-vector $\boldsymbol{w}_\perp$ that is transverse, that is to say, it has vanishing divergence,

$$\boxed{\boldsymbol{\partial} \cdot \boldsymbol{w}_\perp = 0} \ . \tag{30}$$

Here $\boldsymbol{\partial} \equiv \partial/\partial \boldsymbol{x} \equiv \partial_i \equiv \partial/\partial x^i$ is the gradient in flat 3D space. The scalar and vector parts are also known as spin 0 and spin 1, or gradient and curl, or longitudinal and transverse. The scalar part $\boldsymbol{\partial} w_\parallel$ contains 1 degree of freedom, while the vector part $\boldsymbol{w}_\perp$ contains 2 degrees of freedom. Together they account for the 3 degrees of freedom of the vector $\boldsymbol{w}$.

Proof: Take the divergence of equation (29)

$$\boldsymbol{\partial} \cdot \boldsymbol{w} = \partial^2 w_\parallel \ . \tag{31}$$

The operator ∂^2 on the right hand side of equation (31) is the 3D Laplacian. The solution of equation (31) is

$$w_\parallel(\boldsymbol{x}) = -\int \frac{\boldsymbol{\partial}' \cdot \boldsymbol{w}(\boldsymbol{x}')}{|\boldsymbol{x}' - \boldsymbol{x}|} \frac{d^3 x'}{4\pi} \ . \tag{32}$$

The solution (32) is valid subject to boundary conditions that the vector $\boldsymbol{w}$ vanish sufficiently rapidly at infinity. In cosmology, the required boundary conditions, which are set at the Big Bang, are apparently satisfied because fluctuations at the Big Bang were small. Equation (29) then immediately implies that the vector part is $\boldsymbol{w}_\perp = \boldsymbol{w} - \boldsymbol{\partial} w_\parallel$.

6.4.2 Fourier version of the decomposition of a vector in flat 3D space

When the background has some symmetry, it is natural to expand perturbations in eigenmodes of the symmetry. If the background space is flat, then it is translation symmetric. Eigenmodes of the translation operator $\boldsymbol{\partial}$ are Fourier modes.

A function $a(\boldsymbol{x})$ in flat 3D space and its Fourier transform $a(\boldsymbol{k})$ are related by (the disposition of factors of 2π in the following definition follows the convention commonly adopted by cosmologists)

$$a(\boldsymbol{k}) = \int a(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, d^3 x \ , \quad a(\boldsymbol{x}) = \int a(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{d^3 k}{(2\pi)^3} \ . \tag{33}$$

You may not be familiar with the practice of using the same symbol a in both real and Fourier space; but a is the same vector in Hilbert space, with components $a_{\boldsymbol{x}}$ in real space, and $a_{\boldsymbol{k}}$ in Fourier space.

Taking the gradient $\boldsymbol{\partial}$ in real space is equivalent to multiplying by $-i\boldsymbol{k}$ in Fourier space

$$\boldsymbol{\partial} \rightarrow -i\boldsymbol{k} \ . \tag{34}$$

Thus the decomposition (29) of the 3D vector $\boldsymbol{w}$ translates into Fourier space as

$$\boldsymbol{w} = -i\, \boldsymbol{k}\, w_\parallel + \boldsymbol{w}_\perp \tag{35}$$

where the vector part $\boldsymbol{w}_\perp$ satisfies

$$\boldsymbol{k} \cdot \boldsymbol{w}_\perp = 0 \ . \tag{36}$$

In other words, in Fourier space the scalar part $\boldsymbol{\partial} w_\parallel$ of the vector $\boldsymbol{w}$ is the part parallel (longitudinal) to the wavevector $\boldsymbol{k}$, while the vector part $\boldsymbol{w}_\perp$ is the part perpendicular (transverse) to the wavevector $\boldsymbol{k}$.

6.4.3 Decomposition of a tensor in flat 3D space

Similarly, the 9 components of a 3×3 spatial matrix h_{ij} can be decomposed into 3 scalars, 2 vectors, and 1 tensor:

$$h_{ij} = \underbrace{\tfrac{1}{3}\,\delta_{ij}\,\phi}_{\text{scalar}} + \underbrace{\partial_i\partial_j h}_{\text{scalar}} + \underbrace{\varepsilon_{ijk}\,\partial_k \tilde{h}}_{\text{scalar}} + \underbrace{\partial_i h_j}_{\text{vector}} + \underbrace{\partial_j \tilde{h}_i}_{\text{vector}} + \underbrace{h^T_{ij}}_{\text{tensor}} \ . \tag{37}$$

In this context, the term **tensor** signifies a 3×3 matrix h^T_{ij} that is traceless, symmetric, and transverse:

$$h^{T\,i}{}_i = 0 \ , \quad h^T_{ij} = h^T_{ji} \ , \quad \partial_i h^T_{ij} = 0 \ . \tag{38}$$

The transverse-traceless-symmetric matrix h^T_{ij} has two degrees of freedom.

6.5 Flat space background

General relativistic perturbation theory is simplest in the case that the unperturbed background space is Minkowski space. In Cartesian coordinates $x^\mu = \{t, x, y, z\}$, the unperturbed coordinate metric is the Minkowski metric

$$\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu} \ . \tag{39}$$

In this section the tetrad is taken to be orthonormal, and aligned with the unperturbed coordinate axes, so that the unperturbed vierbein is the unit matrix

$$\overset{0}{e}_m{}^\mu = \delta^\mu_m \ . \tag{40}$$

6.5.1 Classification of vierbein perturbations

The aims of this subsection are two-fold. First, decompose perturbations into scalar, vector, and tensor parts. Second, identify the coordinate and tetrad gauge-invariant perturbations.

The vierbein perturbations φ_{mn} decompose into 6 scalars, 4 vectors, and 1 tensor

$$\varphi_{tt} = \underbrace{\psi}_{\text{scalar}} \ , \tag{41a}$$

$$\varphi_{ti} = \underbrace{\partial_i w}_{\text{scalar}} + \underbrace{w_i}_{\text{vector}} \ , \tag{41b}$$

$$\varphi_{it} = \underbrace{\partial_i \tilde{w}}_{\text{scalar}} + \underbrace{\tilde{w}_i}_{\text{vector}} \ , \tag{41c}$$

$$\varphi_{ij} = \underbrace{\delta_{ij}\,\Phi}_{\text{scalar}} + \underbrace{\partial_i\partial_j h}_{\text{scalar}} + \underbrace{\varepsilon_{ijk}\,\partial_k \tilde{h}}_{\text{scalar}} + \underbrace{\partial_i h_j}_{\text{vector}} + \underbrace{\partial_j \tilde{h}_i}_{\text{vector}} + \underbrace{h_{ij}}_{\text{tensor}} \ . \tag{41d}$$

For a single Fourier mode whose wavevector $\boldsymbol{k}$ is taken without loss of generality to lie in the x-direction, equations (41) are

$$\varphi_{mn} = \begin{pmatrix} \psi & \partial_x w & w_y & w_z \\ \partial_x \tilde{w} & \Phi + \partial_x^2 h & \partial_x h_y & \partial_x h_z \\ \tilde{w}_y & \partial_x \tilde{h}_y & \Phi + h_{yy} & h_{yz} + \partial_x \tilde{h} \\ \tilde{w}_z & \partial_x \tilde{h}_z & h_{yz} - \partial_x \tilde{h} & \Phi - h_{yy} \end{pmatrix} \ . \tag{42}$$

To identify coordinate gauge-invariant quantities, it is necessary to consider infinitesimal coordinate gauge transformations (8). The tetrad-frame components ϵ_m of the coordinate shift of the coordinate gauge transformation decompose into 2 scalars and 1 vector

$$\epsilon_m = \{ \underset{\text{scalar}}{\epsilon_t} \ , \quad \underset{\text{scalar}}{\partial_i \epsilon} + \underset{\text{vector}}{\epsilon_i} \} \ . \tag{43}$$

In the flat space background space being considered, and with the usual general relativistic assumption of vanishing torsion, the coordinate gauge transformation (26) of the vierbein perturbation simplifies to

$$\varphi_{mn} \rightarrow \varphi'_{mn} = \varphi_{mn} - \partial_m \epsilon_n \ . \tag{44}$$

In terms of the scalar, vector, and tensor potentials introduced in equations (41), the gauge transformations (44) are

$$\varphi_{tt} \rightarrow \underset{\text{scalar}}{\psi - \partial_t \epsilon_t} \ , \tag{45a}$$

$$\varphi_{ti} \rightarrow \partial_i \underset{\text{scalar}}{(w - \partial_t \epsilon)} + \underset{\text{vector}}{(w_i - \partial_t \epsilon_i)} \ , \tag{45b}$$

$$\varphi_{it} \rightarrow \partial_i \underset{\text{scalar}}{(\tilde{w} - \epsilon_t)} + \underset{\text{vector}}{\tilde{w}_i} \ , \tag{45c}$$

$$\varphi_{ij} \rightarrow \underset{\text{scalar}}{\delta_{ij} \, \Phi} + \underset{\text{scalar}}{\partial_i \partial_j (h - \epsilon)} + \underset{\text{scalar}}{\varepsilon_{ijk} \, \partial_k \tilde{h}} + \underset{\text{vector}}{\partial_i (h_j - \epsilon_j)} + \underset{\text{vector}}{\partial_j \tilde{h}_i} + \underset{\text{tensor}}{h_{ij}} \ . \tag{45d}$$

Equations (45a) imply that under an infinitesimal coordinate gauge transformation

$$\psi \rightarrow \psi - \partial_t \epsilon_t \ , \tag{46a}$$

$$w \rightarrow w - \partial_t \epsilon \ , \tag{46b}$$

$$w_i \rightarrow w_i - \partial_t \epsilon_i \ , \tag{46c}$$

$$\tilde{w} \rightarrow \tilde{w} - \epsilon_t \ , \tag{46d}$$

$$\tilde{w}_i \rightarrow \tilde{w}_i \ , \tag{46e}$$

$$\Phi \rightarrow \Phi \ , \tag{46f}$$

$$h \rightarrow h - \epsilon \ , \tag{46g}$$

$$\tilde{h} \rightarrow \tilde{h} \ , \tag{46h}$$

$$h_i \rightarrow h_i - \epsilon_i \ , \tag{46i}$$

$$\tilde{h}_i \rightarrow \tilde{h}_i \ , \tag{46j}$$

$$h_{ij} \rightarrow h_{ij} \ . \tag{46k}$$

Eliminating the coordinate shift ϵ_m from the transformations (46) yields 12 coordinate gauge-invariant combinations of the potentials

$$\psi - \partial_t \tilde{w} \ , \quad w - \partial_t h \ , \quad w_i - \partial_t h_i \ , \quad \tilde{w}_i \ , \quad \Phi \ , \quad \tilde{h} \ , \quad \tilde{h}_i \ , \quad h_{ij} \ . \tag{47}$$

Physical perturbations are not only coordinate but also tetrad gauge-invariant. A quantity is tetrad gauge-invariant if and only if it depends only on the symmetric part of the vierbein pertubations, not on the antisymmetric part. There are 6 combinations of the coordinate gauge-invariant perturbations (47) that are symmetric, and therefore not only coordinate but also tetrad gauge-invariant. These 6 coordinate and tetrad gauge-invariant perturbations comprise 2 scalars, 1 vector, and 1 tensor

$$\boxed{\underset{\text{scalar}}{\Psi} \equiv \psi - \partial_t(w + \tilde{w} - \partial_t h)} \ , \tag{48a}$$

$$\boxed{\underset{\text{scalar}}{\Phi}} \ , \tag{48b}$$

$$\boxed{\underset{\text{vector}}{W_i} \equiv w_i + \tilde{w}_i - \partial_t(h_i + \tilde{h}_i)} \ , \tag{48c}$$

$$\boxed{\underset{\text{tensor}}{h_{ij}}} \ . \tag{48d}$$

6.5.2 Metric, tetrad connections, and Einstein tensor

This subsection gives expressions in a completely general gauge for perturbed quantities in flat background Minkowski space.

The perturbed coordinate metric $g_{\mu\nu}$, equation (5), is

$$\boxed{g_{tt} = 1 + 2\,\psi} \ , \tag{49a}$$

$$\boxed{g_{ti} = \partial_i(w + \tilde{w}) + (w_i + \tilde{w}_i)} \ , \tag{49b}$$

$$\boxed{g_{ij} = \delta_{ij}\,(-1 + 2\,\Phi) + 2\,\partial_i\partial_j h + \partial_i(h_j + \tilde{h}_j) + \partial_j(h_i + \tilde{h}_i) + 2\,h_{ij}} \ . \tag{49c}$$

The coordinate metric is tetrad gauge-invariant, but not coordinate gauge-invariant.

The perturbed tetrad connections Γ_{kmn} are

$$\boxed{\Gamma_{tit} = \partial_i(\psi - \partial_t\tilde{w}) - \partial_t\tilde{w}_i} \ , \tag{50a}$$

$$\boxed{\Gamma_{tij} = -\,\delta_{ij}\,\partial_t\Phi + \partial_i\partial_j(w - \partial_t h) + \tfrac{1}{2}(\partial_i W_j + \partial_j W_i) - \partial_j\tilde{w}_i - \partial_t h_{ij}} \ , \tag{50b}$$

$$\boxed{\Gamma_{ijt} = -\,\tfrac{1}{2}(\partial_i W_j - \partial_j W_i) + \partial_t(\varepsilon_{ijl}\,\partial_l\tilde{h} - \partial_i\tilde{h}_j + \partial_j\tilde{h}_i)} \ , \tag{50c}$$

$$\boxed{\Gamma_{ijk} = -\,(\delta_{jk}\partial_i - \delta_{ik}\partial_j)\Phi + \partial_k(\varepsilon_{ijl}\,\partial_l\tilde{h} - \partial_i\tilde{h}_j + \partial_j\tilde{h}_i) - \partial_i h_{jk} + \partial_j h_{ik}} \ . \tag{50d}$$

Being purely tetrad-frame quantities, the tetrad connections are automatically coordinate gauge-invariant. However, they are not tetrad gauge-invariant, as is evident from the fact that they depend on antisymmetric parts of the vierbein perturbations φ_{mn}.

One of the advantages of working with tetrads is that tetrad-frame quantities such as the tetrad connections Γ_{kmn} and the tetrad-frame Riemann tensor R_{klmn} are by construction independent of the choice of coordinates, and are therefore automatically coordinate gauge-invariant. In the tetrad formalism, you do not have to work too hard (is that really ever true?) to construct coordinate gauge-invariant combinations of the vierbein perturbations

φ_{mn}: the tetrad-frame connections and Riemann tensor will automatically give you the coordinate gauge-invariant combinations. You can check that in the present case the tetrad connections (50) depend only on, and on all 12 of, the coordinate gauge-invariant combinations (47).

The perturbed tetrad-frame Einstein tensor G_{mn} is

$$\boxed{G_{tt} = \underset{\text{scalar}}{2\,\partial^2\Phi}} \ , \tag{51a}$$

$$\boxed{G_{ti} = \underset{\text{scalar}}{2\,\partial_t\partial_i\Phi} + \underset{\text{vector}}{\tfrac{1}{2}\,\partial^2 W_i}} \ , \tag{51b}$$

$$\boxed{G_{ij} = \underset{\text{scalar}}{2\,\delta_{ij}\,\partial_t^2\Phi} + \underset{\text{scalar}}{(\delta_{ij}\partial^2 - \partial_i\partial_j)(\Psi - \Phi)} + \underset{\text{vector}}{\tfrac{1}{2}\,\partial_t(\partial_i W_j + \partial_j W_i)} - \underset{\text{tensor}}{\Box h_{ij}}} \ , \tag{51c}$$

where $\Box$ is the d'Alembertian, the 4-dimensional wave operator

$$\Box \equiv \partial_m \partial^m = \partial_t^2 - \partial^2 \ . \tag{52}$$

Being a tetrad-frame quantity, the tetrad-frame Einstein tensor is automatically coordinate gauge-invariant. Equations (51) show that the tetrad-frame Einstein tensor is also tetrad gauge-invariant, since it depends only on the tetrad-gauge invariant combinations (48) of the vierbein perturbations. The property that the Einstein tensor is tetrad as well as coordinate gauge-invariant is a feature of empty background space, and does not persist to more general spacetimes, such as the Friedmann-Robertson-Walker spacetime.

In a frame with the wavector $\boldsymbol{k}$ taken along the x-axis, the perturbed Einstein tensor is

$$G_{mn} = \begin{pmatrix} 2\,\partial_x^2\Phi & 2\,\partial_x\partial_t\Phi & \frac{1}{2}\,\partial_x^2 W_y & \frac{1}{2}\,\partial_x^2 W_z \\ 2\,\partial_x\partial_t\Phi & 2\,\partial_t^2\Phi & \frac{1}{2}\,\partial_x\partial_t W_y & \frac{1}{2}\,\partial_x\partial_t W_z \\ \frac{1}{2}\,\partial_x^2 W_y & \frac{1}{2}\,\partial_x\partial_t W_y & 2\,\partial_t^2\Phi + \partial_x^2(\Psi - \Phi) - \Box h_+ & -\Box h_\times \\ \frac{1}{2}\,\partial_x^2 W_z & \frac{1}{2}\,\partial_x\partial_t W_z & -\Box h_\times & 2\,\partial_t^2\Phi + \partial_x^2(\Psi - \Phi) + \Box h_+ \end{pmatrix} \tag{53}$$

where h_+ and $h_\times$ are the two polarizations of gravitational waves, discussed further in §6.5.11,

$$h_+ \equiv h_{yy} = -\,h_{zz} \ , \quad h_\times \equiv h_{yz} = h_{zy} \ . \tag{54}$$

6.5.3 Gauge choices

Since only the 6 tetrad and coordinate gauge-invariant potentials Ψ, Φ, W_i, and h_{ij} have physical significance, it is legitimate to choose a particular **gauge**, a set of conditions on the non-gauge-invariant potentials, arranged to simplify the equations, or to bring out some physical aspect. Three gauges considered below are harmonic gauge (§6.5.5), Newtonian gauge (§6.5.6), and synchronous gauge (§6.5.7).

6.5.4 Einstein equations

The Einstein equations are as usual (units $c = G = 1$)

$$G_{mn} = 8\pi T_{mn} \ . \tag{55}$$

There are 10 Einstein equations, but the Einstein tensor (51) depends on only 6 independent potentials: the two scalars Ψ and Φ, the vector W_i, and the tensor h_{ij}. The system of Einstein equations is thus overcomplete. Why? The answer is that 4 of the Einstein equation enforce conservation of energy-momentum, and can therefore be considered as governing the evolution of the energy-momentum as opposed to being equations for the gravitational potentials. For example, the form of equations (51a) and (51b) for G_{tt} and G_{ti} enforces conservation of energy

$$D_m G^{mt} = 0 \ , \tag{56}$$

while the form of equations (51b) and (51c) for G_{ti} and G_{ij} enforces conservation of momentum

$$D_m G^{mi} = 0 \ . \tag{57}$$

Normally, the equations governing the evolution of the energy-momentum T_a^{mn} of each species a of mass-energy would be set up so as to ensure overall conservation of energy-momentum. If this is done, then the conservation equations (56) and (57) can be regarded as redundant. Since equations (56) and (57) are equations for the time evolution of G_{tt} and G_{ti}, one might think that the Einstein equations for G_{tt} and G_{ti} would become redundant, but this is not quite true. In fact the Einstein equations for G_{tt} and G_{ti} impose constraints that must be satisfied on the initial spatial hypersurface. Conservation of energy-momentum guarantees that those constraints will continue to be satisfied on subsequent spatial hypersurfaces, but still the initial conditions must be arranged to satisfy the constraints. Because the Einstein equations for G_{tt} and G_{ti} must be satisfied as constraints on the initial conditions, but thereafter can be ignored, the equations are called constraint equations. The Einstein equation for G_{tt} is called the energy constraint, or Hamiltonian constraint. The Einstein equations for G_{ti} are called the momentum constraints.

6.5.5 Gravity propagates at the speed of light

The tensor component of the Einstein equations shows that, in a vacuum $T_{mn} = 0$, the tensor perturbations h_{ij} propagate at the speed of light, satisfying the wave equation

$$\Box h_{ij} = 0 \ . \tag{58}$$

The tensor perturbations represent gravitational waves.

It is to be expected that scalar and vector perturbations would also propagate at the speed of light, yet this is not obvious from the form of the Einstein tensor (51). Specifically, there are 4 components of the Einstein tensor (51) that apparently depend only on spatial derivatives, not on time derivatives. The 4 corresponding Einstein equations are

$$
\begin{aligned}
\partial^2 \Phi &= 4\pi T_{tt} \ , && \text{(59a)} \\
\partial^2 W_i &= 16\pi \underset{\text{vector}}{T_{ti}} \ , && \text{(59b)} \\
\partial^2 (\Psi - \Phi) &= -\,8\pi Q_{ij} T^{ij} \ , && \text{(59c)}
\end{aligned}
$$

where Q_{ij} in equation (59c) is the quadrupole operator defined below, equation (93). These conditions must be satisfied everywhere at every instant of time, giving the impression that signals are traveling instantaneously from place to place. As discussed in §6.5.4, the first two equations (59a) and (59b) can be regarded as constraints, as opposed to governing the evolution of the potentials. However, the third equation (59c) is not a constraint.

The fact that all the gravitational potentials do propagate at the speed of light can be demonstrated by choosing a particular gauge, **harmonic gauge**, which can be considered as an analog of the Lorenz (not Lorentz!) gauge of electromagnetism. Harmonic gauge is the set of 4 coordinate conditions

$$
\partial^m (\varphi_{mn} + \varphi_{nm}) - \partial_n \varphi_m{}^m = 0 \ . \tag{60}
$$

The conditions (60) are arranged in a form that is tetrad gauge-invariant (the conditions depend only on the symmetric part of φ_{mn}). The quantities on the left hand side of equations (60) transform under a coordinate gauge transformation, in accordance with (44), as

$$
\partial^m (\varphi_{mn} + \varphi_{nm}) - \partial_n \varphi_m{}^m \rightarrow \partial^m (\varphi_{mn} + \varphi_{nm}) - \partial_n \varphi_m{}^m - \Box \epsilon_n \ . \tag{61}
$$

The change $\Box \epsilon_n$ resulting from the coordinate gauge transformation is the 4-dimensional wave operator $\Box$ acting on the coordinate shift ϵ_n, and indeed the harmonic gauge conditions (60) follow uniquely from the requirements (a) that the change produced by a coordinate gauge transformation be $\Box \epsilon_n$, and (b) that the conditions be tetrad gauge-invariant. The harmonic gauge conditions (60) can be accomplished as a particular solution of the wave equation for the coordinate shift ϵ_n. In terms of the potentials defined by equations (41) and (48), the 4 harmonic gauge conditions (60) are

$$
\begin{aligned}
\partial_t (\Psi + 3\Phi) + \Box (w + \tilde{w} - \partial_t h) &= 0 \ , && \text{(62a)} \\
\partial_t W_i + \Box (h_i + \tilde{h}_i) &= 0 \ , && \text{(62b)} \\
-\Psi + \Phi + \Box h &= 0 \ , && \text{(62c)}
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
-\,4\,\partial_t \Phi &= \Box (w + \tilde{w}) \ , && \text{(63a)} \\
-\,\partial_t W_i &= \Box (h_i + \tilde{h}_i) \ , && \text{(63b)} \\
\Psi - \Phi &= \Box h \ . && \text{(63c)}
\end{aligned}
$$

Substituting equations (63) into the spatial components G_{ij} of the Einstein tensor, equations (51c), shows that in vacuum, $T_{mn} = 0$, the scalar and vector potentials do indeed propagate at the speed of light, just like the tensor potentials. As discussed in §6.5.4, the Einstein equations for G_{tt} and G_{ti} can be regarded as constraint equations as opposed to evolution equations, so it suffices that the spatial Einstein equations constitute wave equations to establish that gravity propagates at the speed of light.

The 4 coordinate gauge conditions (60) still leave 6 tetrad gauge conditions to be chosen at will. The most natural choice of tetrad gauge conditions, in the sense that it leads to the greatest simplification of the tetrad connections Γ_{kmn}, equations (50), is

$$\tilde{w} = \tilde{w}_i = \tilde{h} = \tilde{h}_i = 0 \ . \tag{64}$$

6.5.6 Newtonian gauge

If the unperturbed background is Minkowski space, then the most physical gauge is one in which the 6 perturbations retained coincide with the 6 coordinate and tetrad gauge-invariant perturbations (48). This gauge is called **Newtonian gauge**. I think it should be called Copernican gauge. Even though the solar system is a highly non-linear system, from the perspective of general relativity it is a weakly perturbed gravitating system. Applied to the solar system, Newtonian gauge effectively keeps the coordinates aligned with the classical Sun-centered Copernican coordinate frame. By contrast, the coordinates of synchronous gauge (§6.5.7), which are chosen to follow freely-falling bodies, would quickly collapse or get wound up by orbital motions if applied to the solar system, and would cease to provide a useful description.

Newtonian gauge sets

$$w = \tilde{w} = \tilde{w}_i = h = \tilde{h} = h_i = \tilde{h}_i = 0 \ , \tag{65}$$

so that the retained perturbations are the 6 coordinate and tetrad gauge-invariant perturbations (48)

$$\underset{\text{scalar}}{\Psi} \ = \ \psi \ , \tag{66a}$$

$$\underset{\text{scalar}}{\Phi} \ , \tag{66b}$$

$$\underset{\text{vector}}{W_i} \ = \ w_i \ , \tag{66c}$$

$$\underset{\text{tensor}}{h_{ij}} \ . \tag{66d}$$

In Newtonian gauge, the coordinate 4-velocity u^μ of a person at rest in the tetrad frame is

$$u^\mu = \partial_t x^\mu = e_t{}^\mu = \{1 - \Psi, W_i\} \ . \tag{67}$$

This shows that W_i can be interpreted as a 3-velocity at which the tetrad frame is moving through the coordinates. This is the "dragging of inertial frames" discussed in §6.5.9. The proper acceleration experienced by a person at rest in the tetrad frame, with tetrad 4-velocity $u^m = \{1, 0, 0, 0\}$, is

$$\frac{Du^i}{D\tau} = u^t D_t u^i = u^t \big(\partial_t u^i + \Gamma^i_{tt} u^t \big) = \Gamma^i_{tt} = \partial_i \Psi \ . \tag{68}$$

This shows that the "gravity", or minus the proper acceleration, experienced by a person at rest in the tetrad frame is minus the gradient of the potential Ψ.

The Newtonian metric is

$$ds^2 = (1 + 2\,\Psi)\,dt^2 + 2\,W_i\,dt\,dx^i + (-\,\delta_{ij} + 2\,\delta_{ij}\,\Phi + 2\,h_{ij})\,dx^i\,dx^j\ . \tag{69}$$

Since scalar, vector, and tensor perturbations evolve independently, it is legitimate to consider each in isolation. For example, if one is interested only in scalar perturbations, then it is fine to keep only the scalar potentials Ψ and Φ non-zero. Furthermore, as discussed in §6.5.10, since the difference $\Psi - \Phi$ in scalar potentials is sourced by anisotropic relativistic pressure, which is typically small, it is often a good approximation to set $\Psi = \Phi$.

6.5.7 Synchronous gauge

One of the earliest gauges used in general relativistic perturbation theory, and still (in its conformal version) widely used in cosmology, is **synchronous gauge**. As will be seen below, equations (74) and (75), synchronous gauge effectively chooses a coordinate system and tetrad that is attached to the locally inertial frames of freely falling observers. This is fine as long as the observers move only slightly from their initial positions, but the coordinate system will fail when the system evolves too far, even if, as in the solar system, the gravitational perturbations remain weak and therefore treatable in principle with perturbation theory.

Synchronous gauge sets the time components φ_{mn} with $m = t$ or $n = t$ of the vierbein perturbations to zero

$$\psi = w = \tilde{w} = w_i = \tilde{w}_i = 0\ , \tag{70}$$

and makes the additional tetrad gauge choices

$$\tilde{h} = \tilde{h}_i = 0\ , \tag{71}$$

with the result that the retained perturbations are the spatial perturbations

$$\underset{\text{scalar}}{\Phi}\ ,\quad \underset{\text{scalar}}{h}\ ,\quad \underset{\text{vector}}{h_i}\ ,\quad \underset{\text{tensor}}{h_{ij}}\ . \tag{72}$$

In terms of these spatial perturbations, the gauge-invariant perturbations (48) are

$$\underset{\text{scalar}}{\Psi} = \partial_t^2 h\ , \tag{73a}$$

$$\underset{\text{scalar}}{\Phi}\ , \tag{73b}$$

$$\underset{\text{vector}}{W_i} = -\,\partial_t h_i\ , \tag{73c}$$

$$\underset{\text{tensor}}{h_{ij}}\ . \tag{73d}$$

In synchronous gauge, a person at rest in the tetrad frame has coordinate 4-velocity

$$u^\mu = e_t{}^\mu = \{1, 0, 0, 0\}\ , \tag{74}$$

so that the tetrad rest frame coincides with the coordinate rest frame. Moreover a person at rest in the tetrad frame is freely falling, which follows from the fact that the acceleration experienced by a person at rest in the tetrad frame is zero

$$\frac{Du^k}{D\tau} = u^t \big(\partial_t u^k + \Gamma^k_{tt} u^t \big) = \Gamma^k_{tt} = 0 \ , \tag{75}$$

in which $\partial_t u^k = 0$ because the 4-velocity at rest in the tetrad frame is constant, $u^k = \{1, 0, 0, 0\}$, and $\Gamma^k_{tt} = 0$ from equations (50a) with the synchronous gauge choices (70) and (71).

The synchronous gauge choices (70) and (71) have the effect of setting the tetrad time axis equal to the coordinate time axis, $\boldsymbol{\gamma}_t = \boldsymbol{g}_t$, and of ensuring that the time axis is orthogonal to the spatial axes, $\boldsymbol{\gamma}_t \cdot \boldsymbol{g}_i = 0$ for $i = 1, 2, 3$. These choices bring the coordinate axes $\boldsymbol{g}_\mu$ into ADM form with unit lapse and vanishing shift. The extrinsic curvature K_{ij} is (overdot means $\partial_t \equiv \partial/\partial t$), from equation (50b) with the synchronous gauge conditions (70) and (71),

$$K_{ij} \equiv \Gamma_{itj} = \delta_{ij}\,\dot\Phi + \partial_i \partial_j \dot h + \tfrac{1}{2} (\partial_i \dot h_j + \partial_j \dot h_i) + \dot h_{ij} \tag{76}$$

whose components are simply the time derivatives of the synchronous potentials. The Einstein tensor G_{mn} is, from equations (51) with the synchronous gauge conditions (70) and (71),

$$G_{tt} = \underset{\text{scalar}}{2\,\partial^2 \Phi} \ , \tag{77a}$$

$$G_{ti} = \underset{\text{scalar}}{2\,\partial_i \dot\Phi} - \underset{\text{vector}}{\tfrac{1}{2}\,\partial^2 \dot h_i} \ , \tag{77b}$$

$$G_{ij} = \underset{\text{scalar}}{2\,\delta_{ij}\,\ddot\Phi} + \underset{\text{scalar}}{(\delta_{ij}\partial^2 - \partial_i \partial_j)(\ddot h - \Phi)} - \underset{\text{vector}}{\tfrac{1}{2}\,(\partial_i \ddot h_j + \partial_j \ddot h_i)} - \underset{\text{tensor}}{\Box h_{ij}} \ . \tag{77c}$$

The Einstein equations for G_{tt} and G_{ti}, equations (77a) and (77b), constitute constraint equations, the Hamiltonian and momentum constraints, which must be satisfied by the initial conditions, but which thereafter may be ignored. The Einstein equation for G_{ij}, equation (77c), provides equations for the second time derivative of each of the potentials.

6.5.8 Newtonian potential

Einstein's equations $G_{mn} = 8\pi T_{mn}$ applied to the time-time component G_{tt} of the Einstein tensor, equation (51a), imply Poisson's equation

$$\partial^2 \Phi = 4\pi\rho \tag{78}$$

where ρ is the mass-energy density

$$\rho \equiv T_{tt} \ . \tag{79}$$

The solution of Poisson's equation (78) is

$$\Phi(\boldsymbol{x}) = - \int \frac{\rho(\boldsymbol{x}')\, d^3 x'}{|\boldsymbol{x}' - \boldsymbol{x}|} \ . \tag{80}$$

Consider a finite body, for example the Sun, whose energy-momentum is confined within a certain region. Define the integral of the mass-energy density ρ to be the mass M of the body

$$\int \rho(\boldsymbol{x}') \, d^3x' = M \ . \tag{81}$$

Equation (81) agrees with what the definition of the mass M would be in the non-relativistic limit, and as seen below, equation (84), it is what a distant observer would infer the mass of the body to be based on its gravitational potential Φ far away. Thus equation (81) can be taken as the definition of the mass of the body even when the energy-momentum is relativistic. Choose the origin of the coordinates to be at the center of mass, meaning that

$$\int \boldsymbol{x}' \, \rho(\boldsymbol{x}') \, d^3x' = 0 \ . \tag{82}$$

Consider the potential Φ at a point $\boldsymbol{x}$ far outside the body. Expand the denominator of the integral on the right hand side of equation (80) as a Taylor series in $1/x$ where $x \equiv |\boldsymbol{x}|$

$$\frac{1}{|\boldsymbol{x}' - \boldsymbol{x}|} = \frac{1}{x} \sum_{\ell=0}^{\infty} \left(\frac{x'}{x} \right)^{\ell} P_\ell(\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{x}}') = \frac{1}{x} + \frac{\hat{\boldsymbol{x}} \cdot \boldsymbol{x}'}{x^2} + ... \tag{83}$$

where $P_\ell(\mu)$ are Legendre polynomials. Then

$$\begin{aligned} \Phi(\boldsymbol{x}) &= -\frac{1}{x} \int \rho(\boldsymbol{x}') \, d^3x' - \frac{1}{x^2} \, \hat{\boldsymbol{x}} \cdot \int \boldsymbol{x}' \, \rho(\boldsymbol{x}') \, d^3x' - O(x^{-3}) \\ &= -\frac{M}{x} - O(x^{-3}) \ . \end{aligned} \tag{84}$$

Equation (84) shows that the potential far from a body goes as $\Phi = -M/x$, reproducing the usual Newtonian formula.

6.5.9 Dragging of inertial frames

Einstein's equations applied to the vector part of the time-space component G_{ti} of the Einstein tensor, equation (51b), imply

$$\partial^2 W_i = 16\pi f_i = -\, 16\pi f^i \ , \tag{85}$$

where f^i is the vector part of the energy flux T^{ti}

$$\boldsymbol{f} \equiv f^i \equiv \underset{\text{vector}}{T^{ti}} = -\underset{\text{vector}}{T_{ti}} \ . \tag{86}$$

The solution of equation (85) is

$$W_i(\boldsymbol{x}) = 4 \int \frac{f^i(\boldsymbol{x}') \, d^3x'}{|\boldsymbol{x}' - \boldsymbol{x}|} \ . \tag{87}$$

As in the previous subsection, §6.5.8, consider a finite body, such as the Sun, whose energy-momentum is confined within a certain region. Work in the rest frame of the body

$$\int \boldsymbol{f}(\boldsymbol{x}')\, d^3x' = 0 \ . \tag{88}$$

Define the angular momentum $\boldsymbol{L}$ of the body to be

$$\boldsymbol{L} \equiv \int \boldsymbol{x}' \times \boldsymbol{f}(\boldsymbol{x}')\, d^3x' \ . \tag{89}$$

Equation (89) agrees with what the definition of angular momentum would be in the non-relativistic limit, where the mass-energy flux of a mass density ρ moving at velocity $\boldsymbol{v}$ is $\boldsymbol{f} = \rho\, \boldsymbol{v}$. As will be seen below, the angular momentum (89) is what a distant observer would infer the angular momentum of the body to be based on the potential W_i far away, and equation (89) can be taken to be the definition of the angular momentum of the body even when the energy-momentum is relativistic. As will be proven momentarily, equation (90), the integral $\int x'^i f^j(\boldsymbol{x}')\, d^3x'$ is antisymmetric in ij. To show this, write $f^j = \varepsilon^{jkl}\partial_k \phi_l$ for some potential ϕ_l, which is valid because f^j is the vector (curl) part of the energy flux. Then

$$\int x'^i f^j(\boldsymbol{x}')\, d^3x' = \int x'^i \varepsilon^{jkl} \partial'_k \phi_l(\boldsymbol{x}')\, d^3x' = - \int \varepsilon^{jkl} \phi_l(\boldsymbol{x}') \partial'_k x'^i \, d^3x' = \int \varepsilon^{ijl} \phi_l(\boldsymbol{x}')\, d^3x' \ , \tag{90}$$

where the third expression follows from the second by integration by parts, the surface term vanishing because of the assumption that the energy-momentum of the body is confined within a certain region. Taylor expanding equation (87) using equation (83) gives

$$\begin{aligned} W_i(\boldsymbol{x}) &= \frac{4}{x} \int f^i(\boldsymbol{x}')\, d^3x' + \frac{4}{x^2}\, \hat{x}^j \int x'^j f^i(\boldsymbol{x}')\, d^3x + O(x^{-3}) \\ &= \frac{2}{x^2}\, \hat{x}^j \int \left[x'^j f^i(\boldsymbol{x}') - x'^i f^j(\boldsymbol{x}') \right] d^3x + O(x^{-3}) \\ &= -\frac{2}{x^2}\, \hat{\boldsymbol{x}} \times \boldsymbol{L} + O(x^{-3}) \ . \end{aligned} \tag{91}$$

The vector potential $\boldsymbol{W} \equiv W_i$ points in the direction of rotation, right-handedly about the axis of angular momentum $\boldsymbol{L}$. The potential $\boldsymbol{W}$ can be interpreted as the velocity with which a body of angular momentum $\boldsymbol{L}$ drags inertial frames around it.

6.5.10 Quadrupole pressure

Einstein's equations applied to the part of the Einstein tensor (51c) involving $\Psi - \Phi$ imply

$$\partial^2(\Psi - \Phi) = -\, 8\pi Q_{ij} T^{ij} \ , \tag{92}$$

where Q_{ij} is the quadrupole operator (an integro-differential operator) defined by

$$Q_{ij} \equiv -\, \tfrac{1}{2} \left(\delta_{ij} - 3\, \partial_i \partial_j\, \partial^{-2} \right) \ , \tag{93}$$

with ∂^{-2} the inverse spatial Laplacian operator. In Fourier space, the quadrupole operator is

$$Q_{ij} = -\tfrac{1}{2}\left(\delta_{ij} - 3\,\hat{k}_i\hat{k}_j\right) \ . \tag{94}$$

The quadrupole operator Q_{ij} yields zero when acting on δ^{ij}, and the Laplacian operator ∂^2 when acting on $\partial^i\partial^j$

$$Q_{ij}\delta^{ij} = 0 \ , \quad Q_{ij}\partial^i\partial^j = \partial^2 \ . \tag{95}$$

For a single Fourier mode whose wavevector $\boldsymbol{k}$ is taken to lie in the x-direction, equation (92) is

$$\partial_x^2(\Psi - \Phi) = \tfrac{1}{2}(G_{yy} + G_{zz}) - G_{xx} \ . \tag{96}$$

ASYMPTOTIC BEHAVIOR FAR AWAY?

Equation (92) shows that the source of the difference $\Psi - \Phi$ between the two scalar potentials is the quadrupole pressure. Since the quadrupole pressure is small if either there are no relativistic sources, or any relativistic sources are isotropic, it is often a good approximation to set $\Psi = \Phi$. An exception is where there is a significant anisotropic relativistic component. For example, the energy-momentum tensor of a static electric field is relativistic and anisotropic. However, this is still not enough to ensure that Ψ differs from Φ: if the electric field is spherically symmetric, as in the Reissner-Nordström geometry, then it is still true that $\Psi = \Phi$.

One situation where the difference between Ψ and Φ is appreciable is the case of freely-streaming neutrinos at around the time of recombination in cosmology. The 2008 analysis of the CMB by the WMAP team claims to detect a non-zero value of $\Psi - \Phi$ from a slight shift in the third acoustic peak.

6.5.11 Gravitational waves

The tensor perturbations h_{ij} describe propagating gravitational waves. The two independent components of the tensor perturbations describe two polarizations. The two components are commonly designated h_+ and $h_\times$, equations (54). Gravitational waves induce a quadrupole tidal oscillation transverse to the direction of propagation, and the subscripts $+$ and $\times$ represent the shape of the quadrupole oscillation, as illustrated by Figure 1. The h_+ polarization has a $\cos 2\chi$ shape, while the $h_\times$ polarization has a $\sin 2\chi$ shape, where χ is the azimuthal angle with respect to the y-axis about the direction x of propagation.

Einstein's equations applied to the tensor component of the spatial Einstein tensor (51c) imply that gravitational waves are sourced by the tensor component of the energy-momentum

$$\Box h_{ij} = -\,8\pi \underset{\text{tensor}}{T_{ij}} \ . \tag{97}$$

The solution of the wave equation (97) can be obtained from the Green's function of the d'Alembertian wave operator $\Box$. The Green's function is by definition the solution of the wave equation with a delta-function source. There are retarded solutions, which propagate into the future along the future light cone, and advanced solutions, which propagate into

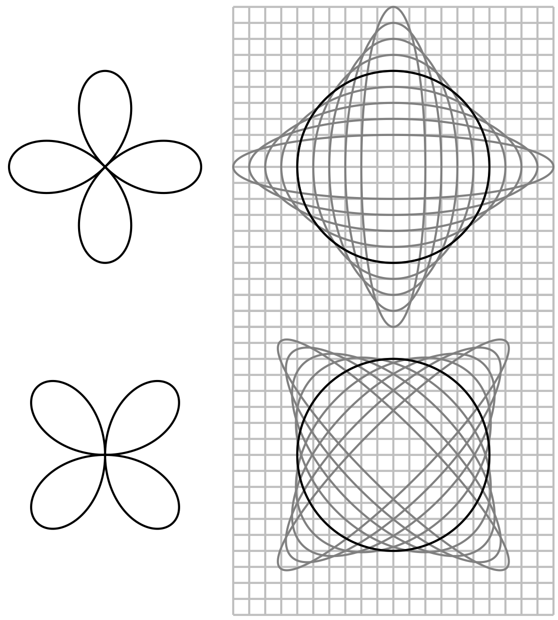

Figure 1: The two polarizations of gravitational waves. The (top) polarization h_+ has a $\cos 2\chi$ shape about the direction of propagation (into the paper), while the (bottom) polarization $h_\times$ has a $\sin 2\chi$ shape. A gravitational wave causes a system of freely falling test masses to oscillate relative to a grid of points a fixed proper distance apart.

the past along the past light cone. In the present case, the solutions of interest are the retarded solutions, since these represent gravitational waves emitted by a source. Because of the time and space translation symmetry of the d'Alembertian, the delta-function source of the Green's function can without loss of generality be taken at the origin $t = x^i = 0$. Thus the Green's function F is the solution of

$$\Box F = \delta^4(0) \ , \tag{98}$$

where $\delta^4(x) \equiv \delta(t)\delta^3(\boldsymbol{x})$ is the 4-dimensional Dirac delta-function. The solution of equation (98) subject to retarded boundary conditions is (a standard exercise in mathematics) the retarded Green's function

$$F = \frac{\delta(x-t)\Theta(t)}{4\pi x} \ , \tag{99}$$

where $x \equiv |\boldsymbol{x}|$ and $\Theta(t)$ is the Heaviside function, $\Theta(t) = 0$ for $t < 0$ and $\Theta(t) = 1$ for $t \geq 0$. The solution of the sourced gravitational wave equation (97) is thus

$$h_{ij}(t, \boldsymbol{x}) = -2 \int \frac{T_{\underset{\text{tensor}}{ij}}(t', \boldsymbol{x}')\, d^3x'}{|\boldsymbol{x}' - \boldsymbol{x}|} \ , \tag{100}$$

where t' is the retarded time

$$t' \equiv t - |\boldsymbol{x}' - \boldsymbol{x}| \ , \tag{101}$$

which lies on the past lightcones of the observer, and is the time at which the source emitted the signal. The solution (100) resembles the solution of Poisson's equation, except that the source is evaluated along the past light cone of the observer.

As in §§6.5.8 and 6.5.9, consider a finite body, whose energy-momentum is confined within a certain region, and which is a source of gravitational waves. The Hulse-Taylor binary pulsar is a fine example. Far from the body, the leading order contribution to the tensor potential h_{ij} is, from the multipole expansion (83),

$$h_{ij}(t, \boldsymbol{x}) = -\frac{2}{x} \int \underset{\text{tensor}}{T_{ij}(t', \boldsymbol{x}')}\, d^3x' \ . \tag{102}$$

The integral (102) is hard to solve in general, but there is a simple solution for gravitational waves whose wavelengths are large compared to the size of the body. To obtain this solution, first consider that conservation of energy-momentum implies that

$$\partial_t^2 T^{tt} - \partial_i \partial_j T^{ji} = \partial_t(\partial_t T^{tt} + \partial_i T^{ti}) - \partial_i(\partial_t T^{ti} + \partial_j T^{ji}) = 0 \ . \tag{103}$$

Multiply by $x^i x^j$ and integrate

$$\int x^i x^j \, \partial_t^2 T^{tt} \, d^3x = \int x^i x^j \, \partial_k \partial_l T^{kl} \, d^3x = \int T^{kl} \partial_k \partial_l (x^i x^j) \, d^3x = 2 \int T^{ij} \, d^3x \ , \tag{104}$$

where the third expression follows from the second by a double integration by parts. For wavelengths that are long compared to the size of the body, the first expression of equations (104) is

$$\int x_i x_j \, \partial_t^2 T^{tt} \, d^3x \approx \partial_t^2 \int x_i x_j \, T^{tt} \, d^3x = \partial_t^2 I_{ij} \tag{105}$$

where I_{ij} is the second moment of the mass

$$I_{ij} \equiv \int x_i x_j \, T^{tt} \, d^3x \ . \tag{106}$$

The tensor (spin-2) part of the energy-momentum is trace-free. The trace-free part $\not{I}_{ij}$ of the second moment I_{ij} is the quadrupole moment of the mass distribution (this definition is conventional, but differs by a factor of 2/3 from what is called the quadrupole moment in spherical harmonics)

$$\not{I}_{ij} = I_{ij} - \tfrac{1}{3}\,\delta_{ij} I^k_k = \int (x_i x_j - \tfrac{1}{3}\,\delta_{ij} x^2) \, T^{tt} \, d^3x \ . \tag{107}$$

Substituting the last expression of equations (104) into equation (102) gives the quadrupole formula for gravitational radiation at wavelengths long compared to the size of the emitting body

$$\boxed{h_{ij}(t, \boldsymbol{x}) = -\frac{1}{x} \underset{\text{tensor}}{\ddot{\not{I}}_{ij}}} \ . \tag{108}$$

If the gravitational wave is moving in the x-direction, then the tensor components of the quadrupole moment $\not{I}_{ij}$ are

$$\not{I}_{+} = \tfrac{1}{2}(I_{yy} - I_{zz}) \ , \quad \not{I}_{\times} = \tfrac{1}{2}(I_{yz} + I_{zy}) \ . \tag{109}$$

6.5.12 Energy-momentum carried by gravitational waves

The gravitational wave equation (58) in empty space appears to describe gravitational waves propagating in a region where the energy-momentum tensor T_{mn} is zero. However, gravitational waves do carry energy-momentum, just as do other kinds of waves, such as electromagnetic waves. The energy-momentum is quadratic in the tensor perturbation h_{ij}, and so vanishes to linear order.

To determine the energy-momentum in gravitational waves, calculate the Einstein tensor G_{mn} to second order, imposing the vacuum conditions that the unperturbed and linear parts of the Einstein tensor vanish

$$\overset{0}{G}_{mn} = \overset{1}{G}_{mn} = 0 \ . \tag{110}$$

The parts of the second-order perturbation that depend on the tensor perturbation h_{ij} are, in a frame where the wavevector $\boldsymbol{k}$ is along the x-axis,

$$\overset{2}{G}_{tt} = -(\partial_t h_{ij})(\partial_t h^{ij}) + \tfrac{1}{4}(\partial_t^2 + \partial_x^2)h^2 \ , \tag{111a}$$

$$\overset{2}{G}_{tx} = -(\partial_t h_{ij})(\partial_x h^{ij}) + \tfrac{1}{2}\partial_t \partial_x h^2 \ , \tag{111b}$$

$$\overset{2}{G}_{xx} = -(\partial_x h_{ij})(\partial_x h^{ij}) + \tfrac{1}{4}(\partial_t^2 + \partial_x^2)h^2 \ , \tag{111c}$$

where

$$h^2 \equiv h_{ij} h^{ij} = 2(h_+^2 + h_\times^2) = 2h_{++}h_{--}. \tag{112}$$

Being tetrad-frame quantities, the expressions (111) are automatically coordinate gauge-invariant, and they are also tetrad gauge-invariant since they depend only on the (coordinate and) tetrad gauge-invariant perturbation h_{ij}. The rightmost set of terms on the right hand side of each of equations (111) are total derivatives (with respect to time t or space x). These terms yield surface terms when integrated over a region, and tend to average to zero when integrated over a region much larger than a wavelength. On the other hand, the leftmost set of terms on the right hand side of each of equations (111) do not average to zero. For example, the terms for G_{tt} and G_{xx} and are negative everywhere, being minus a sum of squares. A negative energy density? The interpretation is that these terms are to be taken over to the right hand side of the Einstein equations, and re-interpreted as the energy-momentum $T^{\rm gw}_{mn}$ in gravitational waves

$$T^{\rm gw}_{tt} \equiv \frac{1}{8\pi}\left[(\partial_t h_{ij})(\partial_t h^{ij}) - \tfrac{1}{4}(\partial_t^2 + \partial_x^2)h^2\right] \ , \tag{113a}$$

$$T^{\rm gw}_{tx} \equiv \frac{1}{8\pi}\left[(\partial_t h_{ij})(\partial_x h^{ij}) - \tfrac{1}{2}\partial_t \partial_x h^2\right] \ , \tag{113b}$$

$$T^{\rm gw}_{xx} \equiv \frac{1}{8\pi}\left[(\partial_x h_{ij})(\partial_x h^{ij}) - \tfrac{1}{4}(\partial_t^2 + \partial_x^2)h^2\right] \ . \tag{113c}$$

The terms involving total derivatives, although they vanish when averaged over a region larger than many wavelengths, ensure that the energy-momentum $T^{\rm gw}_{mn}$ in gravitational waves satisfies conservation of energy-momentum in the flat background space

$$\partial^m T^{\rm gw}_{mn} = 0 \ . \tag{114}$$

Averaged over a region larger than many wavelengths, the energy-momentum in gravitational waves is

$$\boxed{\langle T^{\rm gw}_{mn} \rangle = \frac{1}{8\pi} (\partial_m h_{ij})(\partial_n h^{ij})} \ . \tag{115}$$

Equation (115) may also be written explicitly as a sum over the two polarizations

$$\begin{aligned} \langle T^{\rm gw}_{mn} \rangle &= \frac{1}{4\pi} \left[(\partial_m h_+)(\partial_n h_+) + (\partial_m h_\times)(\partial_n h_\times) \right] \\ &= \frac{1}{8\pi} \left[(\partial_m h_{++})(\partial_n h_{--}) + (\partial_n h_{++})(\partial_m h_{--}) \right] \ . \end{aligned} \tag{116}$$

6.6 Spin

Scalar, vector, and tensor perturbations correspond respectively to perturbations of spin 0, 1, and 2. An object has spin m if it is unchanged by a rotation of $2\pi/m$ about a prescribed axis. In these notes the prescribed axis is taken to be the x-axis. In quantum mechanics, it is usually the z-axis that is selected, but there is nothing fundamental about the choice. A scalar, or spin-0 object, is symmetric about the axis. For non-zero spin, there are two directions of rotation about the axis, two helicities, corresponding to positive and negative spin. An object whose dependence on angle χ about the prescribed axis is $e^{im\chi}$ has spin m.

6.6.1 Spinor tetrad

A systematic way to project objects into spin components is to work in a spinor tetrad. If the prescribed axis of rotation is the x-axis, then the spinor axes $\{\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-\}$ are defined to be complex combinations of the transverse axes $\{\boldsymbol{\gamma}_y, \boldsymbol{\gamma}_z\}$

$$\boxed{\boldsymbol{\gamma}_+ \ \equiv \ \tfrac{1}{\sqrt{2}}(\boldsymbol{\gamma}_y + i\boldsymbol{\gamma}_z)} \ , \tag{117a}$$

$$\boxed{\boldsymbol{\gamma}_- \ \equiv \ \tfrac{1}{\sqrt{2}}(\boldsymbol{\gamma}_y - i\boldsymbol{\gamma}_z)} \ . \tag{117b}$$

The tetrad metric of the spinor tetrad $\{\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_x, \boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-\}$ is

$$\gamma_{mn} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{118}$$

Notice that the spinor axes $\{\boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-\}$ are themselves null, $\boldsymbol{\gamma}_+ \cdot \boldsymbol{\gamma}_+ = \boldsymbol{\gamma}_- = \boldsymbol{\gamma}_- = 0$, whereas their scalar product with each other is non-zero $\boldsymbol{\gamma}_+ \cdot \boldsymbol{\gamma}_- = -1$. The null character of the spinor axes is what makes spin especially well suited to describing fields, such as gravity, that propagate at the speed of light. An even better trick in dealing with fields that propagate at the speed of light is to work in a Newman-Penrose tetrad, §6.7, in which all 4 tetrad axes are taken to be null.

It is a general rule that the spin of a tetrad tensor equals the sum of the +'s and −'s of its covariant indices

$$\boxed{\text{spin} = \text{sum of +'s and −'s of covariant indices}} \ . \tag{119}$$

6.6.2 Spinor components of the Einstein tensor

For a perturbed Minkowski space, in a frame where the wavevector $\boldsymbol{k}$ is taken along the x-axis, the spinor components of the perturbed Einstein tensor G_{mn} are [compare equation (53)]

$$G_{mn} = \begin{pmatrix} 2\,\partial_x^2\Phi & 2\,\partial_x\partial_t\Phi & \frac{1}{2}\,\partial_x^2 W_+ & \frac{1}{2}\,\partial_x^2 W_- \\ 2\,\partial_x\partial_t\Phi & 2\,\partial_t^2\Phi & \frac{1}{2}\,\partial_x\partial_t W_+ & \frac{1}{2}\,\partial_x\partial_t W_- \\ \frac{1}{2}\,\partial_x^2 W_+ & \frac{1}{2}\,\partial_x\partial_t W_+ & -\Box h_{++} & 2\,\partial_t^2\Phi + \partial_x^2(\Psi - \Phi) \\ \frac{1}{2}\,\partial_x^2 W_- & \frac{1}{2}\,\partial_x\partial_t W_- & 2\,\partial_t^2\Phi + \partial_x^2(\Psi - \Phi) & -\Box h_{--} \end{pmatrix} , \tag{120}$$

where $W_\pm$ are the spinor components of the vector perturbation W_i

$$W_\pm = \frac{1}{\sqrt{2}}\,(W_y \pm i\,W_z) \ , \tag{121}$$

and $h_{\pm\pm}$ are the spinor components of the tensor perturbation h_{ij}

$$h_{\pm\pm} = \frac{1}{2}\,(h_{yy} - h_{zz}) \pm i\,h_{yz} = h_+ \pm i\,h_\times \ . \tag{122}$$

The spin +2 and −2 components h_{++} and h_{--} of the tensor perturbation are called the right- and left-handed circular polarizations. The spin +2 and −2 circular polarizations h_{++} and h_{--} have respective shapes $e^{i2\chi}$ and $e^{-i2\chi}$, which may be compared to the $\cos 2\chi$ and $\sin 2\chi$ shapes of the linear polarizations h_+ and $h_\times$.

6.7 Newman-Penrose formalism

The Newman-Penrose formalism provides a particularly powerful way to deal with fields that propagate at the speed of light. The Newman-Penrose formalism adopts a tetrad in which the two axes $\boldsymbol{\gamma}_v$ (outgoing) and $\boldsymbol{\gamma}_u$ (ingoing) along the direction of propagation are chosen to be lightlike, while the two axes $\boldsymbol{\gamma}_+$ and $\boldsymbol{\gamma}_-$ transverse to the direction of propagation are chosen to be spinor axes.

6.7.1 Newman-Penrose tetrad

In a frame where the wavevector $\boldsymbol{k}$ of the perturbation is parallel to the x-direction, the Newman-Penrose tetrad $\{\boldsymbol{\gamma}_v, \boldsymbol{\gamma}_u, \boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-\}$ is defined in terms of an orthonormal tetrad $\{\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_x, \boldsymbol{\gamma}_y, \boldsymbol{\gamma}_z\}$ by

$$\boldsymbol{\gamma}_v \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\gamma}_t + \boldsymbol{\gamma}_x) \ , \tag{123a}$$

$$\boldsymbol{\gamma}_u \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\gamma}_t - \boldsymbol{\gamma}_x) \ , \tag{123b}$$

$$\boldsymbol{\gamma}_+ \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\gamma}_y + i\boldsymbol{\gamma}_z) \ , \tag{123c}$$

$$\boldsymbol{\gamma}_- \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\gamma}_y - i\boldsymbol{\gamma}_z) \ . \tag{123d}$$

All four tetrad axes are null

$$\boldsymbol{\gamma}_v \cdot \boldsymbol{\gamma}_v = \boldsymbol{\gamma}_u \cdot \boldsymbol{\gamma}_u = \boldsymbol{\gamma}_+ \cdot \boldsymbol{\gamma}_+ = \boldsymbol{\gamma}_- \cdot \boldsymbol{\gamma}_- = 0 \ . \tag{124}$$

The metric of the Newman-Penrose tetrad is

$$\gamma_{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{125}$$

A wave $\psi \propto e^{-ik(t-x)}$ moving at the speed of light in the positive x-direction satisfies

$$\partial_v \psi = 0 \ , \tag{126}$$

while a wave $\psi \propto e^{-ik(t+x)}$ moving at the speed of light in the negative x-direction satisfies

$$\partial_u \psi = 0 \ . \tag{127}$$

6.7.2 Newman-Penrose components of the Einstein and Weyl tensors

For a perturbed Minkowski space, with the wavevector $\boldsymbol{k}$ of the perturbation taken along the x-axis, the Newman-Penrose components of the perturbed Einstein tensor are [compare equations (53) and (120)]

$$\begin{aligned}
G_{vv} &= 2\,\partial_v^2 \Phi \ , && (128a)\\
G_{uu} &= 2\,\partial_u^2 \Phi \ , && (128b)\\
G_{vu} &= -\,2\,\partial_v \partial_u \Phi \ , && (128c)\\
G_{+-} &= (\partial_v + \partial_u)^2 \Phi + \tfrac{1}{2}\,(\partial_v - \partial_u)^2 (\Psi - \Phi) \ , && (128d)\\
G_{v\pm} &= \tfrac{1}{4}\,\partial_v (\partial_v - \partial_u) W_\pm \ , && (128e)\\
G_{u\pm} &= \tfrac{1}{4}\,\partial_u (\partial_u - \partial_v) W_\pm \ , && (128f)\\
G_{\pm\pm} &= -\,2\,\partial_v \partial_u h_{\pm\pm} \ . && (128g)
\end{aligned}$$

The Newman-Penrose components of the complexified Weyl tensor are

$$\begin{aligned}
\tilde{C}_{v+v+} &= C_{+2} = -\,\partial_v^2 h_{++} \ , && (129a)\\
\tilde{C}_{vuv+} = -\,\tilde{C}_{+-v+} &= C_{+1} = -\,\tfrac{1}{8}\,(\partial_v - \partial_u)\partial_v W_+ \ , && (129b)\\
\tilde{C}_{vuvu} = \tilde{C}_{+-+-} = -\,\tilde{C}_{vu+-} = -\,\tilde{C}_{v+u-} &= C_0 = -\,\tfrac{1}{12}\,(\partial_v - \partial_u)^2 (\Psi + \Phi) \ , && (129c)\\
\tilde{C}_{uvu-} = -\,\tilde{C}_{-+u-} &= C_{-1} = -\,\tfrac{1}{8}\,(\partial_u - \partial_v)\partial_u W_- \ , && (129d)\\
\tilde{C}_{u-u-} &= C_{-2} = -\,\partial_u^2 h_{--} \ , && (129e)
\end{aligned}$$

Notice that the equations remain unchanged by simultaneous interchange of indices $v \leftrightarrow u$ and $+ \leftrightarrow -$, equivalent to an interchange of tetrad axes

$$\boldsymbol{\gamma}_v \leftrightarrow \boldsymbol{\gamma}_u \ , \quad \boldsymbol{\gamma}_+ \leftrightarrow \boldsymbol{\gamma}_- \ . \tag{130}$$

This means that the equations governing outgoing perturbations of one spin are symmetrically the same as those governing ingoing perturbations of the opposite spin. This symmetry holds in general spacetimes.

6.7.3 Energy-momentum carried by gravitational waves

For a perturbed Minkowski space, with the wavevector $\boldsymbol{k}$ of the perturbation taken along the x-axis, the Newman-Penrose components of the energy-momentum tensor of gravitational waves are [compare equations (113)]

$$T^{\rm gw}_{vv} \equiv \frac{1}{8\pi} \left[(\partial_v h_{ij})(\partial_v h^{ij}) - \tfrac{1}{2}\partial_v^2 h^2 \right] \ , \tag{131a}$$

$$T^{\rm gw}_{vu} \equiv \frac{1}{8\pi} (\partial_v h_{ij})(\partial_u h^{ij}) \ , \tag{131b}$$

$$T^{\rm gw}_{uu} \equiv \frac{1}{8\pi} \left[(\partial_u h_{ij})(\partial_u h^{ij}) - \tfrac{1}{2}\partial_u^2 h^2 \right] \ . \tag{131c}$$

6.7.4 Sachs optical scalars

6.7.5 Raychaudhuri equations

6.8 Cosmological perturbations

Undoubtedly the preeminent application of general relativistic perturbation theory is to cosmology. Fluctations in the temperature and polarization of the Cosmic Microwave Background (CMB) provide an observational window on the Universe at 400,000 years old that, coupled with other astronomical observations, has yielded impressively precise measurements of cosmological parameters.

The theory of cosmological perturbations is based principally on general relativistic perturbation theory coupled to the physics of 5 species of energy-momentum: photons, baryons, non-baryonic cold dark matter, neutrinos, and dark energy.

Dark energy was not important at the time of recombination, where the CMB that we see comes from, but it is important today. If dark energy has a vacuum equation of state, $p = -\rho$, then dark energy does not cluster (vacuum energy density is a constant), but it affects the evolution of the cosmic scale factor, and thereby does affect the clustering of baryons and dark matter today. Moreover the evolution of the gravitational potential along the line-of-sight to the CMB does affect the observed power spectrum of the CMB, the so-called integrated Sachs-Wolfe effect.

Unfortunately, it is beyond the scope of these notes to treat cosmological perturbations in full. For that, consult Scott Dodelson's incomparable text "Modern Cosmology".

6.8.1 An overview of cosmological perturbations

1. **Inflationary initial conditions.** The theory of inflation has been remarkably successful in accounting for many aspects of observational cosmology, even though a fundamental understanding of the inflaton scalar field that supposedly drove inflation is missing. The current paradigm holds that primordial fluctuations were generated by vacuum quantum fluctuations in the inflaton field at the time of inflation. The theory makes the generic predictions that the gravitational potentials generated by

vacuum fluctuations were (a) **Gaussian**, (b) **adiabatic** (meaning that all species of mass-energy fluctuated together, as opposed to in opposition to each other), and (c) **scale-free**, or rather almost scale-free (the fact that inflation came to an end modifies slightly the scale-free character). The three predictions fit the observed power spectrum of the CMB astonishingly well.

2. **Comoving Fourier modes.** The spatial homogeneity of the Friedmann-Robertson-Walker background spacetime means that its perturbations are characterized by Fourier modes of constant comoving wavevector. Each Fourier mode generated by inflation evolved independently, and its wavelength expanded with the Universe.

3. **Scalar, vector, tensor modes.** Spatial isotropy on top of spatial inhomogenity means that the perturbations comprised independently evolving scalar, vector, and tensor modes. Scalar modes dominate the fluctuations of the CMB, and caused the clustering of matter today. Vector modes are usually assumed to vanish, because there is no mechanism to generate the vorticity that sources vector modes, and the expansion of the Universe tends to redshift away any vector modes that might have been present. Inflation generates gravitational waves, which then propagate essentially freely to the present time. Gravitational waves leave an observational imprint in the "B" (curl) mode of polarization of the CMB, whereas scalar modes produce only an "E" (gradient) mode of polarization.

4. **Power spectrum.** The primary quantity measurable from observations is the **power spectrum**, which is the variance of fluctuations of the CMB or of matter (as traced by galaxies, galaxy clusters, the Lyman alpha forest, peculiar velocities, weak lensing, or 21 centimeter observations at high redshift). The statistics of a Gaussian field are completely characterized by its mean and variance. The mean characterizes the unperturbed background, while the variance characterizes the fluctuations. For a 3-dimensional statistically homogeneous and isotropic field, the variance of Fourier modes $\delta_{\boldsymbol{k}}$ defines the power spectrum $P(k)$

$$\langle \delta_{\boldsymbol{k}} \delta_{\boldsymbol{k}'} \rangle = \mathbf{1}_{\boldsymbol{k}\boldsymbol{k}'} P(k) \ , \tag{132}$$

where $\mathbf{1}_{\boldsymbol{k}\boldsymbol{k}'}$ is the unit matrix in the Hilbert space of Fourier modes

$$\mathbf{1}_{\boldsymbol{k}\boldsymbol{k}'} \equiv (2\pi)^3 \delta^3(\boldsymbol{k} + \boldsymbol{k}') \ . \tag{133}$$

The "momentum-conserving" Dirac delta-function in equation (133) is a consequence of spatial translation symmetry. Isotropy implies that the power spectrum $P(k)$ is a function only of the absolute value $k \equiv |\boldsymbol{k}|$ of the wavevector. For a statistically rotation invariant field projected on the sky, such as the CMB, the variance of spherical harmonic modes $\Theta_{\ell m} \equiv \delta T_{\ell m}/T$ defines the power spectrum C_ℓ

$$\langle \Theta_{\ell m} \Theta_{\ell' m'} \rangle = \mathbf{1}_{\ell m, \ell' m'} C_\ell \tag{134}$$

where $\mathbf{1}_{\ell m, \ell' m'}$ is the unit matrix in the Hilbert space of spherical harmonics (distinguish the three usages of δ in this paragraph: δ meaning fluctuation, δ meaning Dirac delta-function, and δ meaning Kronecker delta, as in the following equation)

$$\mathbf{1}_{\ell m, \ell' m'} \equiv \delta_{\ell \ell'} \delta_{m, -m'} \ . \tag{135}$$

Again, the "angular momentum-preserving" condition (??) that $\ell = \ell'$ and $m + m' = 0$ is a consequence of rotational symmetry. The same rotational symmetry implies that the power spectrum C_ℓ is a function only of the harmonic number ℓ, not of the directional harmonic number m.

5. **Reheating.** Early Universe inflation evidently came to an end. It is presumed that the vacuum energy released by the decay of the inflaton field, an event called **reheating**, somehow efficiently produced the matter and radiation fields that we see today. After reheating, the Universe was dominated by relativistic fields, collectively called "radiation". Reheating changed the evolution of the cosmic scale factor from acceleration to deceleration, but is presumed not to have generated additional fluctuations.

6. **Photon-baryon fluid and the sound horizon.** Photon-electron (Thomson) scattering kept photons and baryons tightly coupled to each other, so that they behaved like a relativistic fluid. As long as the radiation density exceeded the baryon density, which remained true up to near the time of recombination, the speed of sound in the photon-baryon fluid was $\sqrt{p/\rho} \approx \sqrt{\frac{1}{3}}$ of the speed of light. Fluctuations with wavelengths outside the sound horizon grew by gravity. As time went by, the sound horizon expanded in comoving radius, and fluctuations thereby came inside the sound horizon. Once inside the sound horizon, sound waves could propagate, which tended to decrease the gravitational potential. However, each individual sound wave itself continued to oscillate, its oscillation amplitude $\delta T/T$ relative to the background temperature T remaining approximately constant. The relativistic suppression of the potential at small scales is responsible for the fact that the power spectrum of matter declines at small scales.

7. **Acoustic peaks in the power spectrum.** The oscillations of the photon-baryon fluid produced the characteristic pattern of peaks and troughs in the CMB power spectrum observed today. The same peaks and troughs occur in the matter power spectrum, but are much less prominent, at a level of about 10% as opposed to the order unity oscillations observed in the CMB power spectrum. For adiabatic fluctuations, the amplitude of the temperature fluctuations follows a pattern $\sim -\cos(k\eta_s)$ where η_s is the comoving sound horizon. The n'th peak occurs at a wavenumber k where $k\eta_s \approx n\pi$. In the observed CMB power spectrum, the relevant value of the sound horizon η_s is its value $\eta_{s,\mathrm{rec}}$ at recombination. Thus the wavenumber k of the first peak of the observed CMB power spectrum occurs where $k\eta_{s,\mathrm{rec}} \approx \pi$. Two competing forces cause a mode to evolve: a gravitational force that amplifies compression, and a restoring pressure force that counteracts compression. When a mode enters the sound horizon for the first time, the compressing gravitational force beats the restoring pressure force, so the first thing that happens is that the mode compresses further. Consequently the first peak is a compression peak. This sets the subsequent pattern: odd peaks are compression peaks, while even peaks are rarefaction peaks. The observed temperature fluctuations of the CMB are produced by a combination of intrinsic temperature fluctuations, Doppler shifts, and gravitational redshifting out of potential wells. The Doppler shift produced by the velocity of a perturbation is 90° out of phase with the temperature fluctuation,

and so tends to fill in the troughs in the power spectrum of the temperature fluctuation. This is the main reasone that the observed CMB power spectrum remains above zero at all scales.

8. **Logarithmic growth of matter fluctuations.** Non-baryonic cold dark matter interacts weakly except by gravity, and is needed to explain the observed clustering of matter in the Universe today in spite of the small amplitude of temperature fluctuations in the CMB. The adjective "cold" refers to the requirement that the dark matter became non-relativistic ($p = 0$) at some early time. If the dark matter is both non-baryonic and cold, then it did not participate in the oscillations of the photon-baryon fluid, and it produced a gravitational potential that did not decay like that of the relativistic photon-baryonic fluid. Consequently, for modes inside the sound horizon, the non-baryonic cold dark matter fluctuations became the dominant source of the gravitational perturbation at an early time, before matter-radiation equality. During this time, the dark matter fluctuations grew logarithmically. The logarithmic growth translates into a logarithmic increase in the amplitude of matter fluctuations at small scales, and is a characteristic signature of non-baryonic cold dark matter. Unfortunately this signature is not readily discernible in the power spectrum of matter today, because of nonlinear clustering.

9. **Epoch of matter-radiation equality.** The density of non-relativistic matter decreases more slowly than the density of relativistic radiation. There came a point where the matter density equaled the radiation density, an epoch called matter-radiation equality, after which the matter density exceeded the radiation density. The observed ratio of the density of matter and radiation (CMB) today require that matter-radiation equality occurred at a redshift of $z \approx 10^4$, a factor of 10 higher in redshift than recombination. After matter-radiation equality, dark matter perturbations grew more rapidly, linearly instead of just logarithmically with cosmic scale factor. Increasing the dark matter density causes matter-radiation equality to occur earlier. The sound horizon at matter-radiation equality corresponds to a scale roughly around the 2.5'th peak in the CMB power spectrum. For adiabatic fluctuations, the way that the temperature and gravitational perturbations interact when a mode first enters the sound horizon means that the temperature oscillation is 5 times larger for modes that enter the horizon well into the radiation-dominated regime versus well into the matter-dominated regime. The effect enhances the amplitude of observed CMB peaks higher than 2.5 relative to those lower than 2.5. The observed relative strengths of the 3rd versus the 2nd peak of the CMB power spectrum provides a measurement of the redshift of matter-radiation equality, and direct evidence for the presence of non-baryonic cold dark matter.

10. **Sound speed.** The density of baryons decreased more slowly than the density of radiation, so that at around recombination the baryon density was becoming comparable to the radiation density. The sound speed $\sqrt{p/\rho}$ depends on the ratio of pressure p, which was essentially entirely that of the photons, to the density ρ, which was produced by both photons and baryons. The sound speed consequently decreased below $\sqrt{\frac{1}{3}}$. Increasing the baryon-to-photon ratio at recombination has several observational ef-

fects on the acoustic peaks of the CMB power spectrum, making it a prime measurable parameter from the CMB. First, an increased baryon fraction increases the gravitational forcing (baryon loading), which enhances the compression (odd) peaks while reducing the rarefaction (even) peaks. Second, increasing the baryon fraction reduces the sound speed, which: (a) decreases the amplitude of the radiation dipole relative to the radiation monopole, so increasing the prominence of the peaks; and (b) reduces the oscillation frequency of the photon-baryon fluid, which shifts the peaks to larger scales. The reduced sound speed also causes an adiabatic reduction of the amplitudes of all modes by the square root of the sound speed, but this effect is degenerate with an overall reduction in the initial amplitudes of modes produced by inflation.

11. **Recombination.** As the temperature cooled below about 3,000 K, electrons combined with hydrogen and helium nuclei into neutral atoms. This drastically reduced the amount of photon-electron scattering, releasing the CMB to propagate almost freely. At the same time, the baryons were released from the photons. Without radiation pressure to support them, fluctuations in the baryons began to grow like the dark matter fluctuations.

12. **Neutrinos.** Probably all three species of neutrino have mass less than 0.3 eV and were therefore relativistic up to and at the time of recombination. Each of the 3 species of neutrino had an abundance comparable to that of photons, and therefore made an important contribution to the relativistic background and its fluctuations. Unlike photons, neutrinos streamed freely, without scattering. The relativistic free-streaming of neutrinos provided the main source of the quadrupole pressure that produces a non-vanishing difference $\Psi - \Phi$ between the scalar potentials. However, the neutrino quadrupole pressure was still only $\sim 10\%$ of the neutrino monopole pressure. To the extent that the neutrino quadrupole pressure can be approximated as negligible, the neutrinos and their fluctuations can be treated the same as photons.

13. **CMB fluctuations.** The CMB fluctuations that we see on the sky today represent a projection of fluctuations on a thin but finite shell at a redshift of about 1100, corresponding to an age of the Universe of about 400,000 yr. The temperature, and the degrees of polarization in two different directions, provide 3 independent observables at each point on the sky. The isotropy of the unperturbed radiation means that it is most natural to measure the fluctuations in spherical harmonics, which are the eigenmodes of the rotation operator. Similarly, it is natural to measure the CMB polarization in spin harmonics.

14. **Matter fluctuations.** After recombination, perturbations in the non-baryonic and baryonic matter grew by gravity, essentially unaffected any longer by photon pressure. If one or more of the neutrino types had a mass small enough to be relativistic but large enough to contribute appreciable density, then its relativistic streaming could have suppressed power in matter fluctuations at small scales, but observations show no evidence of such suppression, which places an upper limit of about an eV on the mass of the most massive neutrino. The matter power spectrum measured from the clustering of galaxies contains acoustic oscillations like the CMB power spectrum, but

because the non-baryonic dark matter dominates the baryons, the oscillations are much smaller.

15. **Intermediate Sachs-Wolfe effect.** Variations in the gravitational potential along the line-of-sight to the CMB affect the CMB power spectrum at large scales. This is called the **intermediate Sachs-Wolfe** (ISW) effect. If matter dominates the background, then the gravitational potential Φ has the property that it remains constant in time for (subhorizon) linear fluctuations, and there is no ISW effect. In practice, ISW effects are produced by at least three distinct causes. First, an early-time ISW effect is produced by the fact that the Universe at recombination still has an appreciable component of radiation, and is not yet wholly matter-dominated. Second, a late-time ISW effect is produced either by curvature or by a cosmological constant. Third, a non-linear ISW effect is produced by non-linear evolution of the potential.

6.9 Flat Friedmann-Robertson-Walker background

For simplicity, these notes will consider only a flat (not closed or open) Friedmann-Robertson-Walker background. The comoving cosmological horizon size at recombination was much smaller than today, and consequently the cosmological density Ω was much closer to 1 at recombination than it is today. Since observations indicate that the Universe today is within 1% of being spatially flat, it is an excellent approximation to treat the Universe at the time of recombination as being spatially flat.

It is convenient to choose the coordinate system $x^\mu = \{\eta, x^i\}$ to consist of conformal time η together with Cartesian comoving coordinates $\boldsymbol{x} \equiv x^i \equiv \{x, y, z\}$. The coordinate metric of the unperturbed background FRW geometry is then

$$ds^2 = a(\eta)^2 \left[d\eta^2 - \left(dx^2 + dy^2 + dz^2 \right) \right] \ . \tag{136}$$

The unperturbed coordinate metric is the conformal Minkowski metric

$$\overset{0}{g}_{\mu\nu} = a(\eta)^2 \eta_{\mu\nu} \ . \tag{137}$$

The tetrad is taken to be orthonormal, and aligned with the unperturbed coordinate axes, so that the unperturbed vierbein is $1/a$ times the unit matrix

$$\overset{0}{e}_m{}^\mu = \frac{1}{a} \, \delta^\mu_m \ . \tag{138}$$

Let overdot denote differentiation with respect to conformal time η,

$$\text{overdot} \equiv \frac{\partial}{\partial \eta} \ , \tag{139}$$

so that for example $\dot{a} \equiv da/d\eta$. The conformal time derivative $\partial/\partial\eta$ is to be distinguished from the directed time derivative $\partial_\eta \equiv e_\eta{}^\mu \, \partial/\partial x^\mu$. Let ∇_i denote the comoving gradient

$$\nabla_i \equiv \frac{\partial}{\partial x^i} \ , \tag{140}$$

which should be distinguished from the directed derivative $\partial_i \equiv e_i{}^\mu \, \partial/\partial x^\mu$.

6.9.1 Classification of vierbein perturbations

The tetrad-frame components φ_{mn} of the vierbein perturbation of the FRW geometry decompose in much the same way as in flat Minkowski case into 6 scalars, 4 vectors, and 1 tensor (the following equations are essentially the same as those (41) for the flat Minkowski background)

$$\varphi_{tt} = \underbrace{\psi}_{\text{scalar}} , \tag{141a}$$

$$\varphi_{ti} = \underbrace{\nabla_i w}_{\text{scalar}} + \underbrace{w_i}_{\text{vector}} , \tag{141b}$$

$$\varphi_{it} = \underbrace{\nabla_i \tilde{w}}_{\text{scalar}} + \underbrace{\tilde{w}_i}_{\text{vector}} , \tag{141c}$$

$$\varphi_{ij} = \underbrace{\delta_{ij}\,\phi}_{\text{scalar}} + \underbrace{\nabla_i\nabla_j h}_{\text{scalar}} + \underbrace{\varepsilon_{ijk}\,\nabla_k \tilde{h}}_{\text{scalar}} + \underbrace{\nabla_i h_j}_{\text{vector}} + \underbrace{\nabla_j \tilde{h}_i}_{\text{vector}} + \underbrace{h_{ij}}_{\text{tensor}} . \tag{141d}$$

The tetrad-frame components ϵ_m of the coordinate shift of the coordinate gauge transformation (8) similarly decompose into 2 scalars and 1 vector (the following equation is essentially the same as that (43) for the flat Minkowski background)

$$\epsilon_m = \{ \underbrace{\epsilon_\eta}_{\text{scalar}} , \ \underbrace{\nabla_i \epsilon}_{\text{scalar}} + \underbrace{\epsilon_i}_{\text{vector}} \} . \tag{142}$$

The vierbein perturbations φ_{mn} transform under a coordinate gauge transformation (8) as, equation (26),

$$\varphi_{tt} \rightarrow \underbrace{\psi - \frac{1}{a}\frac{\partial \epsilon_\eta}{\partial \eta}}_{\text{scalar}} , \tag{143a}$$

$$\varphi_{ti} \rightarrow \nabla_i \underbrace{\left(w - \frac{1}{a}\Big(\frac{\partial}{\partial\eta} - \frac{\dot{a}}{a}\Big)\epsilon \right)}_{\text{scalar}} + \underbrace{\left(w_i - \frac{1}{a}\Big(\frac{\partial}{\partial\eta} - \frac{\dot{a}}{a}\Big)\epsilon_i \right)}_{\text{vector}} , \tag{143b}$$

$$\varphi_{it} \rightarrow \nabla_i \underbrace{\left(\tilde{w} - \frac{1}{a}\epsilon_\eta \right)}_{\text{scalar}} + \underbrace{\tilde{w}_i}_{\text{vector}} , \tag{143c}$$

$$\varphi_{ij} \rightarrow \delta_{ij} \underbrace{\left(\phi + \frac{\dot{a}}{a^2}\epsilon_\eta \right)}_{\text{scalar}} + \nabla_i\nabla_j \underbrace{\left(h - \frac{1}{a}\epsilon \right)}_{\text{scalar}} + \underbrace{\varepsilon_{ijk}\,\nabla_k \tilde{h}}_{\text{scalar}} + \nabla_i \underbrace{\left(h_j - \frac{1}{a}\epsilon_j \right)}_{\text{vector}} + \underbrace{\nabla_j \tilde{h}_i}_{\text{vector}} + \underbrace{h_{ij}}_{\text{tensor}} , \tag{143d}$$

or equivalently

$$\begin{aligned}
\psi &\rightarrow \psi - \frac{1}{a}\frac{\partial \epsilon_\eta}{\partial \eta} \ , && \text{(144a)}\\
w &\rightarrow w - \frac{1}{a}\Big(\frac{\partial}{\partial \eta} - \frac{\dot a}{a}\Big)\epsilon \ , && \text{(144b)}\\
w_i &\rightarrow w_i - \frac{1}{a}\Big(\frac{\partial}{\partial \eta} - \frac{\dot a}{a}\Big)\epsilon_i \ , && \text{(144c)}\\
\tilde w &\rightarrow \tilde w - \frac{1}{a}\epsilon_\eta \ , && \text{(144d)}\\
\tilde w_i &\rightarrow \tilde w_i \ , && \text{(144e)}\\
\phi &\rightarrow \phi + \frac{\dot a}{a^2}\epsilon_\eta \ , && \text{(144f)}\\
h &\rightarrow h - \frac{1}{a}\epsilon \ , && \text{(144g)}\\
\tilde h &\rightarrow \tilde h \ , && \text{(144h)}\\
h_i &\rightarrow h_i - \frac{1}{a}\epsilon_i \ , && \text{(144i)}\\
\tilde h_i &\rightarrow \tilde h_i \ , && \text{(144j)}\\
h_{ij} &\rightarrow h_{ij} \ . && \text{(144k)}
\end{aligned}$$

Eliminating the coordinate shift ϵ_m from the transformations (144) yields 12 coordinate gauge-invariant combinations of the perturbations

$$\psi - \Big(\frac{\partial}{\partial \eta} + \frac{\dot a}{a}\Big)\tilde w \ , \quad w - \frac{\partial h}{\partial \eta} \ , \quad w_i - \frac{\partial h_i}{\partial \eta} \ , \quad \tilde w_i \ , \quad \phi + \frac{\dot a}{a}\tilde w \ , \quad \tilde h \ , \quad \tilde h_i \ , \quad h_{ij} \ . \qquad (145)$$

Six combinations of these coordinate gauge-invariant perturbations depend only on the symmetric part $\varphi_{mn} + \varphi_{nm}$ of the vierbein perturbations, and are therefore tetrad gauge-invariant as well as coordinate gauge-invariant. These 6 coordinate and tetrad gauge-invariant perturbations comprise 2 scalars, 1 vector, and 1 tensor

$$\boxed{\underset{\text{scalar}}{\Psi} \equiv \psi - \Big(\frac{\partial}{\partial \eta} + \frac{\dot a}{a}\Big)\Big(w + \tilde w - \frac{\partial h}{\partial \eta}\Big)} \ , \qquad (146a)$$

$$\boxed{\underset{\text{scalar}}{\Phi} \equiv \phi + \frac{\dot a}{a}\Big(w + \tilde w - \frac{\partial h}{\partial \eta}\Big)} \ , \qquad (146b)$$

$$\boxed{\underset{\text{vector}}{W_i} \equiv w_i + \tilde w_i - \frac{\partial}{\partial \eta}(h_i + \tilde h_i)} \ , \qquad (146c)$$

$$\boxed{\underset{\text{tensor}}{h_{ij}}} \ . \qquad (146d)$$

6.9.2 Metric, tetrad connections, and Einstein tensor

This subsection gives expressions in a completely general gauge for perturbed quantities in the flat Friedmann-Robertson-Walker background geometry.

The perturbed coordinate metric $g_{\mu\nu}$ is

$$\begin{aligned} g_{\eta\eta} &= a^2(1+2\,\psi)\ , && (147a)\\ g_{\eta i} &= a^2\big[\nabla_i(w+\tilde{w})+(w_i+\tilde{w}_i)\big]\ , && (147b)\\ g_{ij} &= a^2\big[(-1+2\,\phi)\delta_{ij}+2\,\nabla_i\nabla_j h+\nabla_i(h_j+\tilde{h}_j)+\nabla_j(h_i+\tilde{h}_i)+2\,h_{ij}\big]\ . && (147c)\end{aligned}$$

The coordinate metric is tetrad gauge-invariant, but not coordinate gauge-invariant.

The perturbed tetrad connections Γ_{kmn} are

$$\begin{aligned} \Gamma_{\eta i\eta} &= \frac{1}{a}\left[\nabla_i\left(\psi-\Big(\frac{\partial}{\partial\eta}+\frac{\dot{a}}{a}\Big)\tilde{w}\right)-\Big(\frac{\partial}{\partial\eta}+\frac{\dot{a}}{a}\Big)\tilde{w}_i\right]\ , && (148a)\\ \Gamma_{\eta ij} &= \frac{1}{a}\left[\Big(\frac{\dot{a}}{a}-F\Big)\delta_{ij}+\nabla_i\nabla_j\Big(w-\frac{\partial h}{\partial\eta}\Big)+\frac{1}{2}(\nabla_iW_j+\nabla_jW_i)-\nabla_j\tilde{w}_i+\frac{\partial h_{ij}}{\partial\eta}\right]\ , && (148b)\\ \Gamma_{ij\eta} &= \frac{1}{a}\left[-\frac{1}{2}(\nabla_iW_j-\nabla_jW_i)+\frac{\partial}{\partial\eta}(\varepsilon_{ijl}\,\nabla_l\tilde{h}-\nabla_i\tilde{h}_j+\nabla_j\tilde{h}_i)\right]\ , && (148c)\\ \Gamma_{ijk} &= \frac{1}{a}\Bigg[(\delta_{ik}\nabla_j-\delta_{jk}\nabla_i)\Big(\phi+\frac{\dot{a}}{a}\tilde{w}\Big)+\frac{\dot{a}}{a}(\delta_{ik}\delta_{jl}-\delta_{jk}\delta_{il})\tilde{w}_l\\ &\qquad +\nabla_k(\varepsilon_{ijl}\,\nabla_l\tilde{h}-\nabla_i\tilde{h}_j+\nabla_j\tilde{h}_i)-\nabla_ih_{jk}+\nabla_jh_{ik}\Bigg]\ , && (148d)\end{aligned}$$

where F is defined by

$$F\equiv\dot{\phi}+\frac{\dot{a}}{a}\psi\ . \quad (149)$$

Being purely tetrad-frame quantities, the tetrad connections are automatically coordinate gauge-invariant, but they are not tetrad gauge-invariant. The quantity F defined by equation (155) is not coordinate gauge-invariant, but the combination $\dot{a}/a^2-F/a$ that appears in the expression (148b) for $\Gamma_{\eta ij}$ is coordinate and tetrad gauge-invariant.

The perturbed tetrad-frame Einstein tensor G_{mn} is

$$\begin{aligned} G_{\eta\eta} &= \frac{1}{a^2}\left[3\,\frac{\dot{a}^2}{a^2}-6\,\frac{\dot{a}}{a}F+2\,\nabla^2\Phi\right]\ , && (150a)\\ G_{\eta i} &= \frac{1}{a^2}\left[2\,\nabla_i\left(F+\Big(\frac{\ddot{a}}{a}-2\,\frac{\dot{a}^2}{a^2}\Big)\tilde{w}\right)+\frac{1}{2}\nabla^2W_i+2\,\Big(\frac{\ddot{a}}{a}-2\,\frac{\dot{a}^2}{a^2}\Big)\tilde{w}_i\right]\ , && (150b)\\ G_{ij} &= \frac{1}{a^2}\Bigg[\left(-2\,\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+2\,\Big(\frac{\partial}{\partial\eta}+2\,\frac{\dot{a}}{a}\Big)F+2\,\Big(\frac{\ddot{a}}{a}-2\,\frac{\dot{a}^2}{a^2}\Big)\psi\right)\delta_{ij}\\ &\qquad +(\delta_{ij}\nabla^2-\nabla_i\nabla_j)(\Psi-\Phi)\\ &\qquad +\frac{1}{2}\Big(\frac{\partial}{\partial\eta}+2\,\frac{\dot{a}}{a}\Big)(\nabla_iW_j+\nabla_jW_i)-\Big(\frac{\partial^2}{\partial\eta^2}+2\,\frac{\dot{a}}{a}\frac{\partial}{\partial\eta}-\nabla^2\Big)h_{ij}\Bigg]\ . && (150c)\end{aligned}$$

Being tetrad-frame quantities, all components of the tetrad-frame Einstein tensor are automatically coordinate gauge-invariant. The time-time $G_{\eta\eta}$ and space-space G_{ij} components are not only coordinate but also tetrad gauge-invariant. Specifically, the quantities

$3(\dot{a}^2/a^4) - 6(\dot{a}/a^3)F$ on the right hand side of equation (150a) for $G_{\eta\eta}$, and the coefficient of δ_{ij} on the right hand side of equation (150c) for G_{ij}, are coordinate and tetrad gauge-invariant. However, the time-space components $G_{\eta i}$ are not tetrad gauge-invariant, as is evident from the fact that equation (150b) involves the non-tetrad-gauge-invariant perturbations $\tilde{w}$ and $\tilde{w}_i$. Physically, under a tetrad boost by a velocity v of linear order, the time-space components $G_{\eta i}$ change by first order v, but $G_{\eta\eta}$ and G_{ij} change only to second order v^2. Thus to linear order, only $G_{\eta i}$ changes under a tetrad boost. Note that $G_{\eta i}$ changes under a tetrad boost ($\tilde{w}$ and $\tilde{w}_i$), but not under a tetrad rotation ($\tilde{h}$ and $\tilde{h}_i$).

6.9.3 Comoving Fourier modes

Since the unperturbed Friedmann-Robertson-Walker spacetime is spatially homogeneous and isotropic, it is natural to work in comoving Fourier modes. Comoving Fourier modes have the key property that they evolve independently of each other, as long as perturbations remain linear. Equations in Fourier space are obtained by replacing the comoving spatial gradient ∇_i by $-i$ times the comoving wavevector k_i

$$\nabla_i \rightarrow -ik_i \ . \tag{151}$$

By this means, the partial differential equations governing the evolution of perturbations become ordinary differential equations.

6.9.4 Conformal Newtonian gauge

Conformal Newtonian gauge sets

$$w = \tilde{w} = \tilde{w}_i = h = \tilde{h} = h_i = \tilde{h}_i = 0 \ , \tag{152}$$

so that the retained perturbations are the 6 coordinate and tetrad gauge-invariant perturbations (146)

$$\underset{\text{scalar}}{\Psi} = \psi \ , \tag{153a}$$

$$\underset{\text{scalar}}{\Phi} = \phi \ , \tag{153b}$$

$$\underset{\text{vector}}{W_i} = w_i \ , \tag{153c}$$

$$\underset{\text{tensor}}{h_{ij}} \ . \tag{153d}$$

In conformal Newtonian gauge, the scalar perturbations of the Einstein equations are the energy density, energy flux, monopole pressure, and quadrupole pressure equations

$$-3\,\frac{\dot{a}}{a}F - k^2\Phi = 4\pi a^2 \overset{1}{T}{}^{\eta\eta} \ , \tag{154a}$$

$$ikF = 4\pi a^2\, \hat{k}_i\, \overset{1}{T}{}^{\eta i} \ , \tag{154b}$$

$$\dot{F} + 2\,\frac{\dot{a}}{a}\,F + \left(\frac{\ddot{a}}{a} - 2\,\frac{\dot{a}^2}{a^2}\right)\Psi - \frac{k^2}{3}(\Psi - \Phi) = \frac{4}{3}\pi a^2\, \delta_{ij}\, \overset{1}{T}{}^{ij} \ , \tag{154c}$$

$$-k^2(\Psi - \Phi) = 4\pi a^2 \left(\tfrac{3}{2}\,\hat{k}_i\hat{k}_j - \tfrac{1}{2}\,\delta_{ij}\right) \overset{1}{T}{}^{ij} \ , \tag{154d}$$

where

$$F \equiv \dot{\Phi} + \frac{\dot{a}}{a}\Psi \ . \tag{155}$$

Only 1 of the 3 equations (154a)–(154c) is independent.

6.10 Energy-momentum

To obtain a closed set of equations, the Einstein equations must be coupled to equations governing each species of energy-momentum: photons, baryons, non-baryonic cold dark matter, neutrinos, and dark energy.

6.10.1 Phase space distribution

Each species of energy-momentum is described by a phase-space distribution, a function $f(\eta, \boldsymbol{x}, \boldsymbol{p})$ of conformal time η, comoving position $\boldsymbol{x}$, and proper momentum $\boldsymbol{p}$, which describes the number dN of particles in an interval $a^3 d^3x\, d^3p/(2\pi\hbar)^3$ of phase space

$$dN(\eta, \boldsymbol{x}, \boldsymbol{p}) = f(\eta, \boldsymbol{x}, \boldsymbol{p}) \, \frac{g\, a^3 d^3x\, d^3p}{(2\pi\hbar)^3} \tag{156}$$

with g being the number of spin states of the particle. The interval $a^3 d^3x\, d^3p/(2\pi\hbar)^3$ of phase space is constant in the unperturbed background, since the proper momentum redshifts as $\boldsymbol{p} \propto a^{-1}$. The (proper) energy-momentum 4-vector p^m in the tetrad frame is

$$p^m \equiv e^m{}_\mu \frac{dx^\mu}{d\lambda} = \{E, \boldsymbol{p}\} = \{E, p^i\} \ , \tag{157}$$

where λ is an affine parameter. The proper energy E and momentum $p \equiv |\boldsymbol{p}|$ for a particle of mass m are related by

$$E^2 - p^2 = m^2 \ . \tag{158}$$

The (proper) tetrad-frame 3-velocity $\boldsymbol{v}$ of a particle is related to its energy E and momentum $\boldsymbol{p}$ in the usual way

$$\boldsymbol{v} \equiv \frac{\boldsymbol{p}}{E} \ . \tag{159}$$

The tetrad-frame components of the energy-momentum tensor T^{mn} of any particular species are integrals over the phase space distributions f

$$T^{\eta\eta} = \int f\, E \, \frac{g\, d^3p}{(2\pi\hbar)^3} \ , \tag{160a}$$

$$T^{\eta i} = \int f\, E v^i \, \frac{g\, d^3p}{(2\pi\hbar)^3} \ , \tag{160b}$$

$$T^{ij} = \int f\, E v^i v^j \, \frac{g\, d^3p}{(2\pi\hbar)^3} \ . \tag{160c}$$

6.10.2 Boltzmann equation

The evolution of each species of energy-momentum is described by the general relativistic Boltzmann equation

$$\frac{df}{d\lambda} = C[f] \ , \tag{161}$$

where $C[f]$ is a collision term. The left hand side of the Boltzmann equation (161) is

$$\frac{df}{d\lambda} = p^m \partial_m f + \frac{dp^i}{d\lambda}\frac{\partial f}{\partial p^i} = E\partial_\eta f + p^i \partial_i f + \frac{d\hat{\boldsymbol{p}}}{d\lambda} \cdot \frac{\partial f}{\partial \hat{\boldsymbol{p}}} + \frac{dp}{d\lambda}\frac{\partial f}{\partial p} \ . \tag{162}$$

Both $d\hat{\boldsymbol{p}}/d\lambda$ and $\partial f/\partial \hat{\boldsymbol{p}}$ vanish in the unperturbed background, so $d\hat{\boldsymbol{p}}/d\lambda \cdot \partial f/\partial \hat{\boldsymbol{p}}$ is of second order, and can be neglected to linear order, so that

$$\frac{df}{d\lambda} = E\partial_\eta f + p^i \partial_i f + \frac{dp}{d\lambda}\frac{\partial f}{\partial p} \ . \tag{163}$$

The expression (163) for the left hand side $df/d\lambda$ of the Boltzmann equation involves $dp/d\lambda$, which in free-fall is determined by the usual geodesic equation

$$\frac{dp^k}{d\lambda} + \Gamma^k_{mn} p^m p^n = 0 \ . \tag{164}$$

Since $E^2 - p^2 = m^2$, it follows that the equation of motion for the magnitude p of the tetrad-frame momentum is related to the equation of motion for the tetrad-frame energy E by

$$p\frac{dp}{d\lambda} = E\frac{dE}{d\lambda} \ . \tag{165}$$

In conformal Newtonian gauge, and for scalar fluctuations, the equation of motion for the tetrad-frame energy $E \equiv p^\eta$ is

$$\frac{dE}{d\lambda} = -\Gamma^\eta_{mn} p^m p^n = -\Gamma_{\eta i \eta} p^i E - \Gamma_{\eta ij} p^i p^j = \frac{1}{a}\left[-Ep^i \nabla_i \Psi - p^2\left(\frac{\dot{a}}{a} - F\right)\right] \ . \tag{166}$$

From this it follows that

$$\frac{dp}{d\lambda} = -\frac{Ep}{a}\left(\frac{\dot{a}}{a} - \frac{\dot{a}}{a}\Psi - \dot{\Phi} + \frac{E\hat{p}^i}{p}\nabla_i \Psi\right) \ . \tag{167}$$

In practice, the integration variable used to evolve equations is the conformal time η, not the affine parameter λ. The relation between conformal time η and affine parameter λ is, in conformal Newtonian gauge,

$$\frac{d\eta}{d\lambda} = \frac{dx^\eta}{d\lambda} = e_m{}^\eta p^m = (\delta^\eta_m - \phi_m{}^\eta)\overset{0}{e}_n{}^\mu p^m = \frac{E}{a}(1 - \Psi) \ . \tag{168}$$

With conformal time η as the integration variable, the equation of motion (167) for the magnitude p of the tetrad-frame momentum becomes, to linear order,

$$\frac{dp}{d\eta} = -\, p \left(\frac{\dot{a}}{a} - \dot{\Phi} + \frac{E \hat{p}^i}{p} \nabla_i \Psi \right) \ . \tag{169}$$

With respect to conformal time η, the left hand side (163) of the Boltzmann equation becomes

$$\boxed{\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + v^i \nabla_i f - \left(\frac{\dot{a}}{a} - \dot{\Phi} + \frac{E \hat{p}^i}{p} \nabla_i \Psi \right) \frac{\partial f}{\partial \ln p}} \ . \tag{170}$$

To unperturbed order, the Boltzmann equation (161) is

$$\frac{d\overset{0}{f}}{d\eta} = \frac{a}{E} \, C[\overset{0}{f}] \ , \tag{171}$$

where $C[\overset{0}{f}]$ is the unperturbed collision term, the factor a/E coming from $d\eta/d\lambda = E/a$ to unperturbed order, equation (171). The collision term $C[\overset{0}{f}]$ is non-zero if particles are being created or destroyed in the unperturbed background, as occurs for example during electron-positron annihilation, nucleosynthesis, or recombination. However, the most common situation is that particles in the unperturbed background are in thermodynamic equilibrium, and being neither created nor destroyed, in which case

$$C[\overset{0}{f}] = 0 \ . \tag{172}$$

The left hand side of the unperturbed Boltzmann equation (171) is, from equation (170),

$$\frac{d\overset{0}{f}}{d\eta} = \frac{\partial \overset{0}{f}}{\partial \eta} - \frac{\dot{a}}{a} \frac{\partial \overset{0}{f}}{\partial \ln p} \ . \tag{173}$$

The second term on the right hand side of equation (173) simply reflects the fact that the proper momentum p redshifts as $p \propto 1/a$ as the Universe expands, a statement that is true for both massive and massless particles.

Subtracting off the unperturbed part of the Boltzmann equation gives the perturbation of the Boltzmann equation

$$\frac{d\overset{1}{f}}{d\eta} = \frac{a}{E} \left(\Psi \, C[\overset{0}{f}] + C[\overset{1}{f}] \right) \ , \tag{174}$$

with left hand side

$$\boxed{\frac{d\overset{1}{f}}{d\eta} = \frac{\partial \overset{1}{f}}{\partial \eta} + v^i \nabla_i \overset{1}{f} - \frac{\dot{a}}{a} \frac{\partial \overset{1}{f}}{\partial \ln p} - \left(-\, \dot{\Phi} + \frac{E \hat{p}^i}{p} \nabla_i \Psi \right) \frac{\partial \overset{0}{f}}{\partial \ln p}} \ . \tag{175}$$

6.10.3 Non-baryonic cold dark matter

Non-baryonic cold dark matter, subscripted c, is by assumption non-relativistic and collisionless. The unperturbed mean density is ρ_c. The overdensity δ_c and velocity $\bar{\boldsymbol{v}}_c$ are defined by

$$T_c^{\eta\eta} \equiv \int f_c \, m_c \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \equiv \rho_c = \bar{\rho}_c(1+\delta_c) \ , \tag{176a}$$

$$T_c^{\eta i} \equiv \int f_c \, m_c \, v_c^i \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \equiv \rho_c \, \bar{v}^i \ . \tag{176b}$$

$$\overset{1}{T}{}_c^{\eta\eta} \equiv \int \overset{1}{f}_c \, m_c \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \equiv \rho_c \, \delta_c \ , \tag{177a}$$

$$\overset{1}{T}{}_c^{\eta i} \equiv \int \overset{1}{f}_c \, m_c \, v_c^i \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \equiv \rho_c \, \bar{v}^i \ . \tag{177b}$$

The pressure is of order v_c^2, and can be neglected to linear order.

Integrating the Boltzmann equation gives

$$0 = \int \frac{df_c}{d\eta} m_c \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \tag{178}$$

$$= \frac{\partial}{\partial \eta} \int f_c m_c \, \frac{g_c \, d^3 p_c}{(2\pi)^3} + \nabla_i \int f_c m_c v_c^i \, \frac{g_c \, d^3 p_c}{(2\pi)^3} - \int \left(\frac{\dot{a}}{a} - \dot{\Phi} \right) \frac{\partial f}{\partial \ln p} m_c \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \tag{179}$$

$$= \frac{\partial \rho_c (1+\delta_c)}{\partial \eta} + \nabla_i \rho_c \bar{v}^i + 3 \rho_c \left(\frac{\dot{a}}{a} - \dot{\Phi} \right) \ . \tag{180}$$

$$0 = \int \frac{df_c}{d\eta} m_c v_c^i \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \tag{181}$$

$$= \frac{\partial}{\partial \eta} \int f_c m_c v_c^i \, \frac{g_c \, d^3 p_c}{(2\pi)^3} + \nabla_j \int f_c m_c v_c^i v_c^j \, \frac{g_c \, d^3 p_c}{(2\pi)^3} - \int \frac{E \hat{p}^j}{p} \nabla_j \Psi \frac{\partial f}{\partial \ln p} m_c v^i \, \frac{g_c \, d^3 p_c}{(2\pi)^3} \tag{182}$$

$$= \frac{\partial \rho_c \bar{v}^i}{\partial \eta} + \rho_c \nabla_i \Psi \ . \tag{183}$$

The scalar part of the velocity is, in Fourier space,

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_c = - i \, v_c \ . \tag{184}$$

Non-baryonic cold dark matter is collisionless, so the collision term in the Boltzmann equation is zero, $C[f_c] = 0$, and the dark matter satisfies the Boltzmann equation

$$\frac{df_c}{d\eta} = 0 \ . \tag{185}$$

$$\dot{\delta}_c + \nabla_i v_c^i - 3\dot{\Phi} = 0 \ , \tag{186a}$$

$$\dot{\boldsymbol{v}}_c + \frac{\dot{a}}{a} \bar{\boldsymbol{v}}_c + \boldsymbol{\nabla} \Psi = 0 \ . \tag{186b}$$

6.10.4 Photons

In the unperturbed background, the photons have a blackbody distribution with temperature $T(\eta)$. Define Θ to be the photon temperature fluctuation

$$\Theta(\eta, \boldsymbol{x}, \boldsymbol{p}) \equiv \frac{\delta T(\eta, \boldsymbol{x}, \boldsymbol{p})}{T(\eta)} \ . \tag{187}$$

In the unperturbed background, the photon phase space distribution is

$$\overset{0}{f}_\gamma = \frac{1}{e^{p/T} - 1} \ . \tag{188}$$

Since $p \propto T \propto 1/a$, the unperturbed phase space distribution is constant in time as well as in space. The definition $\Theta \equiv \delta T/T = \delta \ln T$ of the photon perturbation is to be interpreted as meaning that the perturbation to the phase space distribution is of photons is

$$\overset{1}{f}_\gamma = \frac{\partial \overset{0}{f}_\gamma}{\partial \ln T} \, \delta \ln T = \frac{\partial \overset{0}{f}_\gamma}{\partial \ln T} \Theta \ , \tag{189}$$

in which it follows from equation (188) that

$$\frac{\partial \overset{0}{f}_\gamma}{\partial \ln T} = \overset{0}{f}_\gamma (1 + \overset{0}{f}_\gamma) \frac{p}{T} \ , \tag{190}$$

which is constant in time and space.

The photon distribution is modified by photon-electron (Thomson) scattering. Since the electrons are non-relativistic, to linear order collisions change the photon momentum but not the photon energy. As a consequence, the temperature fluctuation is a function $\Theta(\eta, \boldsymbol{x}, \hat{\boldsymbol{p}})$ only of the direction $\hat{\boldsymbol{p}}$ of the photon momentum $\boldsymbol{p}$, not of its magnitude, the energy p.

Scalar fluctuations are those that are rotationally symmetric about the wavevector $\boldsymbol{k}$. Expand in spherical harmonics, keeping only scalar terms, those that are rotationally symmetric about the wavevector $\boldsymbol{k}$

$$\Theta(\eta, \boldsymbol{k}, \hat{\boldsymbol{p}}) = \sum_{\ell=0}^{\infty} i^\ell (2\ell + 1) \Theta_\ell(\eta, \boldsymbol{k}) P_\ell(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{p}}) \tag{191}$$

where P_ℓ are Legendre polynomials.

Perturbations to the photon energy-momentum tensor involve integrals (160) over the perturbed phase space distribution of the form, where $F(\hat{\boldsymbol{p}})$ is some arbitrary function of the momentum direction $\hat{\boldsymbol{p}}$,

$$\int \overset{1}{f}_\gamma \, p \, F(\hat{\boldsymbol{p}}) \frac{2 \, d^3 p}{(2\pi\hbar)^3} = \int \frac{\partial \overset{0}{f}_\gamma}{\partial \ln T} \, p \, \frac{2 \, 4\pi p^2 dp}{(2\pi\hbar)^3} \int \Theta \, F(\hat{\boldsymbol{p}}) \, \frac{do_{\boldsymbol{p}}}{4\pi} = 4\rho_\gamma \int \Theta \, F(\hat{\boldsymbol{p}}) \, \frac{do_{\boldsymbol{p}}}{4\pi} \ , \tag{192}$$

in which the last expression is true because

$$\int \frac{\partial \overset{0}{f}_\gamma}{\partial \ln T}\, p\, \frac{2\, 4\pi p^2 dp}{(2\pi\hbar)^3} = 4 \int \overset{0}{f}_\gamma\, p\, \frac{2\, 4\pi p^2 dp}{(2\pi\hbar)^3} = 4\rho_\gamma \ , \tag{193}$$

which follows from $\partial \overset{0}{f}_\gamma / \partial \ln T = -\, \partial \overset{0}{f}_\gamma / \partial \ln p$ and an integration by parts. The perturbation of the photon energy density, energy flux, monopole pressure, and quadrupole pressure are

$$\overset{1}{T}{}^{\eta\eta}_\gamma = 4\, \rho_\gamma\, \Theta_0 \ , \tag{194a}$$

$$\hat{k}_i\, \overset{1}{T}{}^{\eta i}_\gamma = -\, i\, 4\, \rho_\gamma\, \Theta_1 \ , \tag{194b}$$

$$\tfrac{1}{3}\, \delta_{ij}\, \overset{1}{T}{}^{ij}_\gamma = \tfrac{4}{3}\, \rho_\gamma\, \Theta_0 \ , \tag{194c}$$

$$\left(\tfrac{3}{2}\, \hat{k}_i \hat{k}_j - \tfrac{1}{2}\, \delta_{ij} \right) \overset{1}{T}{}^{ij}_\gamma = -\, 4\, \rho_\gamma\, \Theta_2 \ . \tag{194d}$$

The Boltzmann equation in terms can be recast as a Boltzmann equation for Θ through

$$\frac{d \overset{1}{f}_\gamma}{d\eta} = \frac{\partial \overset{0}{f}_\gamma}{\partial \ln T} \frac{d\Theta}{d\eta} \ . \tag{195}$$

The left hand side (175) of the perturbation to the Boltzmann equation becomes

$$\frac{d\Theta}{d\eta} = \dot{\Theta} + \hat{p}^i\, \nabla_i \Theta - \dot{\Phi} + \frac{E \hat{p}^i}{p} \nabla_i \Psi \ . \tag{196}$$

6.11 A simplest set of assumptions

1. Tight-coupling approximation, that baryons and photons are tightly coupled to each other. This keeps the photons isotropic, so the only two moments are the monopole and dipole.
2. Baryons negligible, so sound speed is $\sqrt{\frac{1}{3}}$.
3. Negligible (Silk) damping from photon-electron (Thomson) scattering.
4. Negligible neutrino quadrupole pressure, so $\Psi = \Phi$.

6.11.1 Equations for the simplest set of assumptions

Radiation:

$$\dot{\Theta}_0 + k\, \Theta_1 = \dot{\Phi} \ , \tag{197a}$$

$$\dot{\Theta}_1 - \frac{k}{3}\, \Theta_0 = \frac{k}{3}\, \dot{\Phi} \ . \tag{197b}$$

Non-baryonic cold dark matter:

$$\dot{\delta}_c - k\, v_c = 3\, \dot{\Phi} \ , \tag{198a}$$

$$\dot{v}_c + \frac{\dot{a}}{a}\, v_c = -\, k\, \Phi \ . \tag{198b}$$

Einstein energy equation:

$$-k^2\Phi - 3\,\frac{\dot{a}}{a}\,F = 4\pi G a^2(\rho_c\delta_c + 4\rho_\gamma\Theta_0) \tag{199}$$

where

$$F \equiv \dot{\Phi} + \frac{\dot{a}}{a}\,\Phi\ . \tag{200}$$

In place of the Einstein energy equation, it is sometimes convenient to use the Einstein momentum equation

$$kF = 4\pi G a^2(\rho_c v_c - 4\rho_\gamma\Theta_1)\ , \tag{201}$$

which is not an independent equation. Exercise: Show explicitly the relation between the Einstein momentum equation and the other equations.

6.11.2 Generic behavior of non-baryonic cold dark matter

Combining equations (198) for the dark matter overdensity and velocity gives

$$\left(\frac{d^2}{d\eta^2} + \frac{\dot{a}}{a}\frac{d}{d\eta}\right)(\delta_c - 3\Phi) = -k^2\Phi \tag{202}$$

In absence of driving potential, $\Phi = 0$, the dark matter velocity redshifts as $\boldsymbol{v}_c \propto 1/a$.

6.11.3 Generic behavior of radiation

Combining equations (197) for the radiation monopole and dipole gives

$$\left(3\frac{d^2}{d\eta^2} + k^2\right)(\Theta_0 - \Phi) = -\,2\,k^2\Phi\ . \tag{203}$$

Equation (203) says that $\Theta_0 - \Phi$ oscillates about -2Φ.

In the absence of a driving potential Φ, the radiation oscillates as $\Theta_0 \propto e^{\pm i\omega\eta}$ with frequency $\omega = \sqrt{\frac{1}{3}}\,k$. In other words, the solutions are sound waves, moving at the sound speed

$$c_s = \frac{\omega}{k} = \sqrt{\frac{1}{3}}\ . \tag{204}$$

6.11.4 Unperturbed background

To calculate the evolution of perturbations to the CMB, need to consider both radiation and matter-dominated phases.

$$\rho_m \propto a^{-3}\ , \quad \rho_\gamma \propto a^{-4}\ . \tag{205}$$

The Hubble parameter is defined in the usual way to be

$$H \equiv \frac{1}{a}\frac{da}{dt} = \frac{\dot{a}}{a^2} \tag{206}$$

in which overdot represents differentiation with respect to conformal time, $\dot{a} \equiv da/d\eta$. The Friedmann equations for the background imply that the Hubble constant for a universe dominated by matter and radiation is

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\gamma) = \frac{H_{\rm eq}^2}{2}\left(\frac{a_{\rm eq}^3}{a^3} + \frac{a_{\rm eq}^4}{a^4}\right) \tag{207}$$

where $a_{\rm eq}$ and $H_{\rm eq}$ are the cosmic scale factor and the Hubble parameter at the time of matter-radiation equality, $\rho_m = \rho_\gamma$.

The comoving horizon distance η is defined to be the comoving distance that light travels starting from zero expansion. This is

$$\eta = \int_0^a \frac{da}{a^2 H} = \frac{2\sqrt{2}}{a_{\rm eq} H_{\rm eq}}\left(\sqrt{1 + \frac{a}{a_{\rm eq}}} - 1\right) = \frac{2\sqrt{2}}{a_{\rm eq} H_{\rm eq}}\left(\frac{a/a_{\rm eq}}{1 + \sqrt{1 + a/a_{\rm eq}}}\right) . \tag{208}$$

In the radiation- and matter-dominated regimes respectively, the comoving horizon distance η is

$$\eta = \begin{cases} \dfrac{\sqrt{2}}{a_{\rm eq} H_{\rm eq}}\left(\dfrac{a}{a_{\rm eq}}\right) & \propto a \quad \text{radiation-dominated ,} \\ \dfrac{2\sqrt{2}}{a_{\rm eq} H_{\rm eq}}\left(\dfrac{a}{a_{\rm eq}}\right)^{1/2} & \propto a^{1/2} \quad \text{matter-dominated .} \end{cases} \tag{209}$$

The ratio of the comoving horizon distance η to the comoving cosmological horizon distance $1/(aH)$ is

$$\eta a H = \frac{2\sqrt{1 + a/a_{\rm eq}}}{1 + \sqrt{1 + a/a_{\rm eq}}} , \tag{210}$$

which is evidently a number of order unity, varying between 1 in the radiation-dominated regime $a \to 0$, and 2 in the matter-dominated regime $a \to \infty$.

6.11.5 Regimes

FROM THIS POINT NOTES ARE STILL UNDER CONSTRUCTION. IN PARTICULAR, MANY MINUS SIGNS ARE WRONG.

1. Superhorizon scales, §6.11.6.
2. Radiation dominates.
3. Radiation dominates the unperturbed background, but matter dominates the fluctuations.
4. Matter dominates, before recombination.
5. Matter dominates, after recombination.

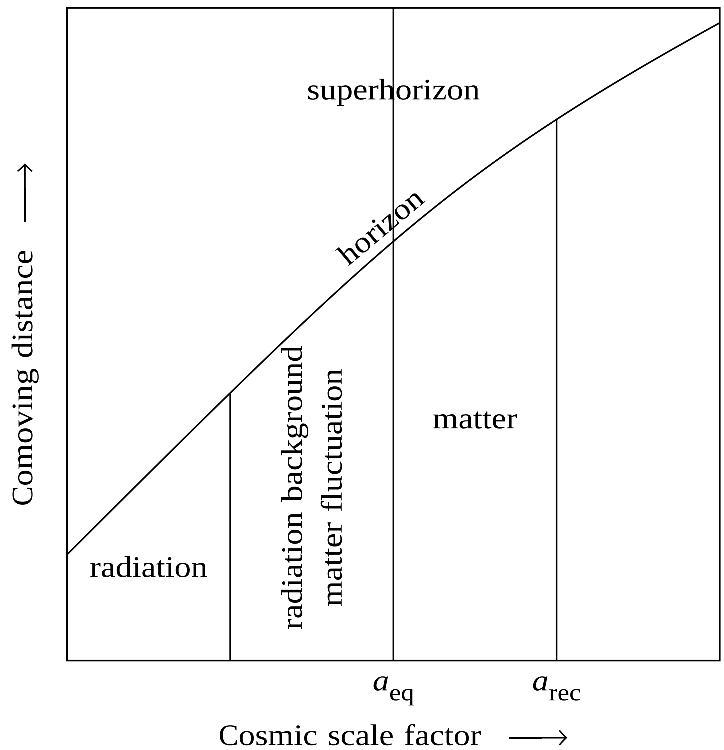

Figure 2: Various regimes in the evolution of fluctuations. The line increasing diagonally from bottom left to top right is the comoving horizon distance η. Above this line are superhorizon fluctuations, whose comoving wavelengths exceed the horizon distance, while below the line are subhorizon fluctuations, whose comoving wavelengths are less than the horizon distance. The vertical line at cosmic scale factor $a_{\rm eq}$ marks the moment of matter-radiation equality. Before matter-radiation equality (to the left), the background mass-energy is dominated by radiation, while after matter-radiation equality (to the right), the background mass-energy is dominated by matter.

6.11.6 Superhorizon scales

At sufficiently early times, any mode is outside the horizon, $k\eta < 1$. In the superhorizon limit $k\eta \ll 1$, the evolution equations (197)–(201) reduce to

$$\dot\Theta_0 = -\dot\Phi \ , \tag{211a}$$

$$\dot\delta_c = -3\dot\Phi \ , \tag{211b}$$

$$3\frac{\dot a}{a}F = 4\pi G a^2(\rho_c\delta_c + 4\rho_\gamma\Theta_0) \ . \tag{211c}$$

The first two of these equations evidently imply

$$\Theta_0 = -\Phi + \text{constant} \ , \tag{212a}$$

$$\delta_c = -3\Phi + \text{constant} \ . \tag{212b}$$

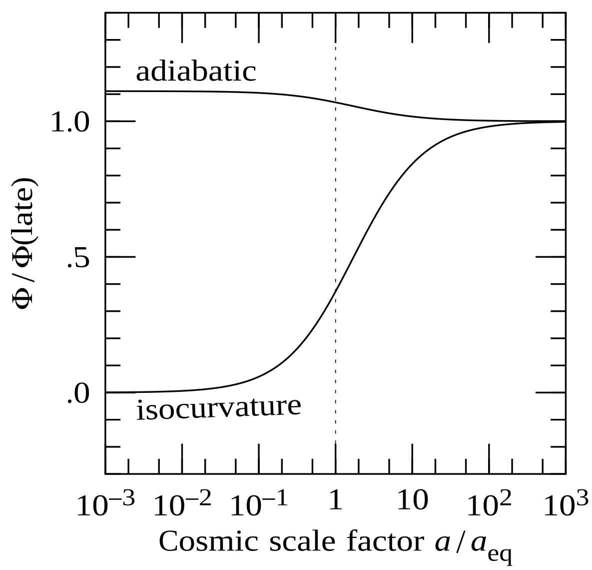

Figure 3: Evolution of the scalar potential Φ at superhorizon scales, from radiation-dominated to matter-dominated.

The constants are set by initial conditions. There are adiabatic and isocurvature initial conditions. Inflation generically produces adiabatic fluctuations, in which radiation and matter fluctuate together

$$\delta_c(0) = 3\,\Theta_0(0) = \tfrac{3}{2}\,\Phi(0)\ . \tag{213}$$

Isocurvature initial conditions are defined by

$$\Phi(0) = 0\ . \tag{214}$$

Plugging the solutions (212) into the Einstein energy equation (211c) gives the first order differential equation

$$2a(1+a)\Phi' + (6+5a)\Phi = C_0 + C_1 a\ , \tag{215}$$

where the constants C_0 and C_1 are

$$C_0 = 4\Phi(0) + 4\Theta_0(0)\ , \quad C_1 = 3\Phi(0) + \delta_c(0)\ . \tag{216}$$

Subject to the condition that Φ remains finite at $a \to 0$, the adiabatic and isocurvature solutions to equation (215) are, in units $a_{\rm eq} = 1$,

$$\Phi_{\rm ad} = \frac{\Phi(0)}{10}\left(9 + \frac{2}{a} - \frac{8}{a^2} - \frac{16}{a^3} + \frac{16\sqrt{1+a}}{a^3}\right) = \frac{\Phi(0)}{10}\frac{\left[3(14+9a) + (38+9a)\sqrt{1+a}\right]}{\left(1+\sqrt{1+a}\right)^3}\ , \tag{217a}$$

$$\Phi_{\rm iso} = \frac{8\,\Phi'(0)}{5}\left(1 - \frac{2}{a} + \frac{8}{a^2} - \frac{16}{a^3} + \frac{16\sqrt{1+a}}{a^3}\right) = \frac{8\,\Phi'(0)}{5}\frac{a\left(6+a+\sqrt{1+a}\right)}{\left(1+\sqrt{1+a}\right)^4}\ , \tag{217b}$$

in which the last expressions in each case are written in a form that is numerically well-behaved for all a. For adiabatic fluctuations

$$\delta_c = 3\Theta_0 = 3\left(\tfrac{3}{2}\Phi(0) - \Phi_{\rm ad}\right) \ , \tag{218}$$

while for isocurvature fluctuations

$$\delta_c = 8\Phi'(0) - 3\Phi_{\rm iso} \ , \quad \Theta_0 = -\,\Phi_{\rm iso} \ . \tag{219}$$

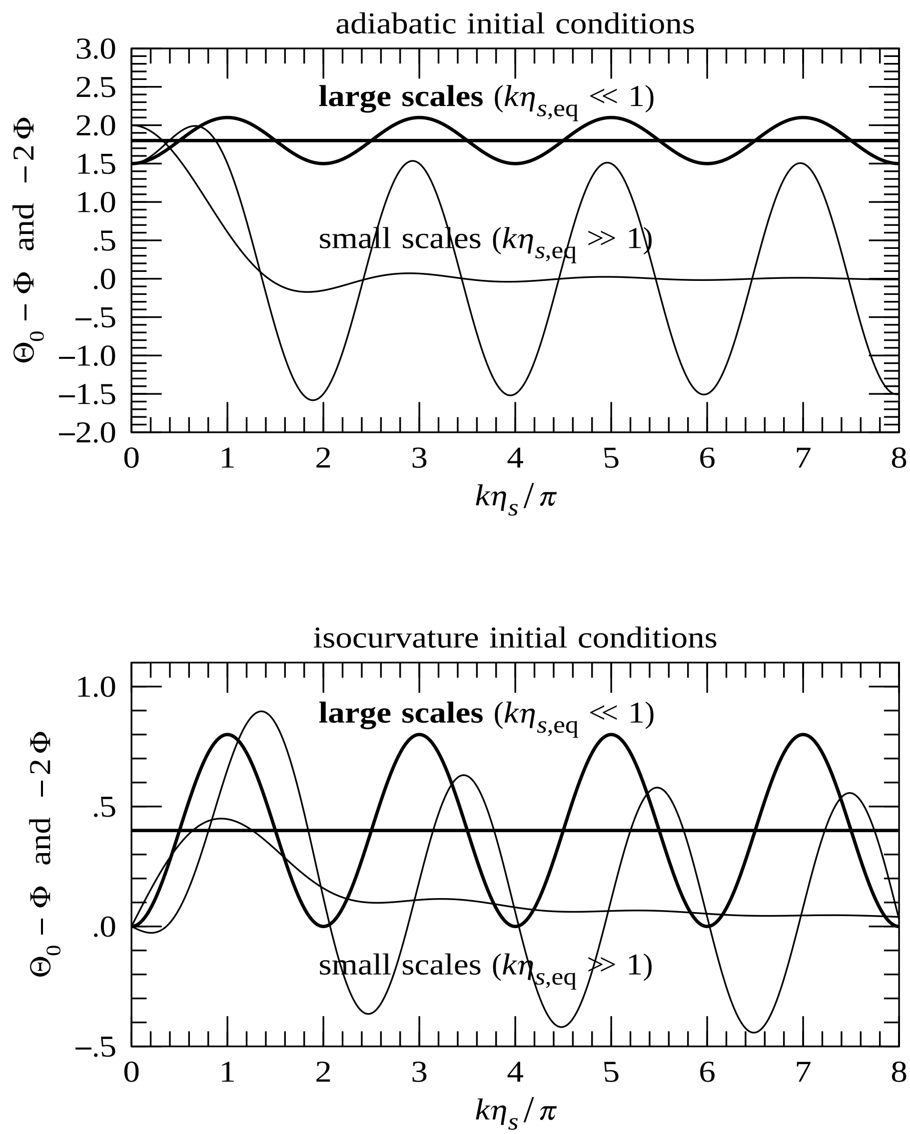

Figure 4: The difference $\Theta_0 - \Phi$ between the radiation monopole and the Newtonian scalar potential Φ oscillates about -2Φ.

6.11.7 Adiabatic: Part 1

For adiabatic initial conditions, fluctuations that enter the horizon before matter-radiation equality, $k\eta_{\rm eq} \gg 1$, are dominated by radiation. In the regime where radiation dominates both the unperturbed energy and its fluctuations, the relevant equations are

$$\begin{aligned} \dot\Theta_0 + k\Theta_1 &= -\dot\Phi \ , && (220a) \\ k^2\Phi + 3\frac{\dot a}{a}F &= 16\pi G a^2 \rho_\gamma \Theta_0 \ , && (220b) \\ kF &= -\,16\pi G a^2 \rho_\gamma \Theta_1 \ , && (220c) \end{aligned}$$

in which, because it simplifies the mathematics, the Einstein momentum equation is used as a substitute for the radiation dipole equation. Inserting Θ_0 and Θ_1 from the Einstein energy and momentum equations (220b) and (220c) into the radiation monopole equation (220a) gives a second order differential equation for the potential Φ

$$\ddot\Phi + \frac{4}{\eta}\dot\Phi + \frac{k^2}{3}\Phi = 0 \ . \tag{221}$$

The equation describes damped sound waves moving at sound speed $\sqrt{\frac{1}{3}}$ times the speed of light. The growing and decaying solutions are

$$\Phi_{\rm grow} = \frac{3j_1(\alpha)}{\alpha} = \frac{3(\sin\alpha - \alpha\cos\alpha)}{\alpha^3} \ , \quad \Phi_{\rm decay} = \frac{j_{-2}(\alpha)}{\alpha} = \frac{\cos\alpha + \alpha\sin\alpha}{\alpha^3} \ , \tag{222}$$

where

$$\alpha \equiv \frac{k\eta}{\sqrt{3}} = \sqrt{\frac{2}{3}}\frac{ka}{a_{\rm eq}^2 H_{\rm eq}} \tag{223}$$

and $j_l(\alpha) \equiv J_{l+1/2}(\alpha)$ is a spherical Bessel function. The physically relevant solution that satisfies adiabatic initial conditions, remaining finite as $\eta \to 0$, is the growing solution

$$\Phi = \Phi(0)\,\Phi_{\rm grow} \ . \tag{224}$$

The solution shows that, after a mode enters the horizon, $k\eta \gg 1$, the scalar potential Φ oscillates with an envelope that decays as a^{-2}. Physically, relativistically propagating sound waves tend to suppress the gravitational potential Φ.

For the growing solution (224), the radiation monopole Θ_0 and dipole Θ_1 are

$$\begin{aligned} \Theta_0 &= \Phi(0)\frac{3}{\alpha^3}\left[-(1-\alpha^2)\sin\alpha + \alpha\left(1-\tfrac{1}{2}\alpha^2\right)\cos\alpha\right] \ , && (225a) \\ \Theta_1 &= \Phi(0)\frac{\sqrt{3}}{\alpha^2}\left[(1-\alpha^2)\sin\alpha - \alpha\cos\alpha\right] \ . && (225b) \end{aligned}$$

After a mode is well inside the horizon, $k\eta \gg 1$, the monopole and dipole oscillate with amplitudes $\frac{3}{2}\Phi(0)$ and $\sqrt{3}\,\Phi(0)$ respectively

$$\left.\begin{aligned} \Theta_0 &= -\tfrac{3}{2}\Phi(0)\,\cos\alpha \\ \Theta_1 &= -\sqrt{3}\,\Phi(0)\,\sin\alpha \end{aligned}\right\} \text{ for } \alpha \gg 1 \ . \tag{226}$$

The dark matter fluctuations are driven by the gravitational potential of the radiation. The solution of the dark matter equation (202) driven by the potential (224) and satisfying adiabatic initial conditions is

$$\delta_c + 3\Phi = 9\Phi(0)\left(\gamma - \frac{1}{2} + \ln\alpha - \mathrm{Ci}\,\alpha + \frac{\sin\alpha}{\alpha}\right) \tag{227}$$

where $\gamma \equiv 0.5772...$ is Euler's constant, $\mathrm{Ci} \equiv \int_\infty^\alpha \cos x\, dx/x$ is the cosine integral, and the potential Φ is given by the growing mode solution (224). Once the mode is well inside the horizon, $k\eta \gg 1$, the dark matter density δ_c evolves as

$$\delta_c = 9\Phi(0)\left(\gamma - \frac{1}{2} + \ln\alpha\right) \quad \text{for } \alpha \gg 1\ , \tag{228}$$

which grows logarithmically. This logarithmic growth translates into a logarithmic increase in the amplitude of matter fluctuations at small scales, and is a characteristic signature of non-baryonic cold dark matter.

6.11.8 Adiabatic: Part 2

After a mode enters the horizon during the radiation-dominated regime, the radiation fluctuation Θ_0 oscillates, but the non-baryonic cold dark matter fluctuation δ_c grows logarithmically, equation (228). For adiabatic initial conditions the initial amplitudes of dark matter and radiation fluctuations are comparable, but because the dark matter fluctuation grows faster than the radiation fluctuation, the dark matter fluctuation in due course dominates the radiation fluctuation, and this necessarily occurs before matter-radiation equality. That is, the density fluctuation is dominated by the matter

$$\rho_c\, \delta_c > 4\, \rho_\gamma\, \Theta_0\ , \tag{229}$$

even though the mean density is still dominated by radiation

$$\rho_c < \rho_\gamma\ . \tag{230}$$

At this point the gravitational potential Φ starts to be driven by the matter rather than by radiation.

Regard the Einstein energy equation (199) as giving δ_c, and the Einstein momentum equation (201) as giving v_c. Insert these into the dark matter density equation (198a) and eliminate the $\dot{\Theta}_0$ terms using the radiation monopole equation (197a). The result is, in units $a_{\mathrm{eq}} = 1$,

$$2(1+a)a^2\Phi'' + (6+7a)a\Phi' - 2\Phi + 4\Theta_0 = 0\ , \tag{231}$$

where prime $'$ denotes differentiation with respect to cosmic scale factor, d/da. Equation (231) is valid in both matter and radiation-dominated regimes, and for any combination of matter and radiation. As remarked in §6.11.3, if the potential Φ is slowly varying, then the radiation monopole Θ_0 oscillates about an average value of Φ,

$$\langle\Theta_0\rangle = \Phi\ . \tag{232}$$

Inserting this cycle-averaged value of Θ_0 into (231) gives the Meszaros differential equation

$$2(1+a)a^2\Phi'' + (6+7a)a\Phi' + 2\Phi = 0 \ . \tag{233}$$

The solutions of Meszaros' differential equation (233) are

$$\Phi = \frac{3H_{\rm eq}^2}{4k^2}\frac{\delta_c}{a} \tag{234}$$

where δ_c is a linear combination of growing and decaying solutions, in units $a_{\rm eq} = 1$,

$$\delta_{\rm grow} = 1 + \frac{3}{2}a \ , \quad \delta_{\rm decay} = \Big(1 + \frac{3}{2}a\Big)\ln\Big(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\Big) - 3\sqrt{1+a} \tag{235}$$

In the radiation-dominated regime

$$\delta_{\rm grow} = 1 \ , \quad \delta_{\rm decay} = -\ln(a/4) - 3 \quad \text{for } a \ll 1 \ . \tag{236}$$

For modes that enter the horizon well before matter-radiation equality, match smoothly to solutions

$$\delta_c = C_1\,\delta_{\rm grow} + C_2\,\delta_{\rm decay} \tag{237}$$

where

$$C_1 = 9\,\Phi(0)\left[\gamma - \frac{7}{2} + \ln\Big(\sqrt{\frac{2}{3}}\frac{4k}{H_{\rm eq}}\Big)\right] \ , \quad C_2 = -\,9\,\Phi(0) \ . \tag{238}$$

Meanwhile,

$$2(1+a)a\Phi'' + (8+9a)\Phi' + 2\Big(1 + \frac{2ka}{3H_{\rm eq}^2}\Big)\Phi - \delta_c = 0 \tag{239}$$

6.11.9 Isocurvature fluctuations

For isocurvature initial conditions,

$$\delta_c = 8\Phi'(0)\delta_{\rm grow} \tag{240}$$

In the radiation-dominated regime, $a \ll 1$, this equation reduces to

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + k^2\Phi = \frac{3H_{\rm eq}^2}{4a}\,\delta_c \tag{241}$$

This looks the same

6.12 General Friedmann-Robertson-Walker background

Work in conformal time η, and isotropic comoving coordinates

$$ds^2 = a(\eta)^2\left[d\eta^2 - \frac{1}{\left(1+\frac{1}{4}\kappa r^2\right)^2}\left(dx^2+dy^2+dz^2\right)\right] \tag{242}$$

where $r \equiv \sqrt{x^2+y^2+z^2}$.