PHYS 5770 Gravitational Theory Spring 2008. Problem Set 1. Due Tue 29 Jan

1. Lorentz transformation

Relative to person A (unprimed frame), person B (primed frame) moves at velocity v along the x-axis. Derive the form of the Lorentz transformation between the coordinates (t, x, y, z) of a 4-vector in A's frame and the corresponding coordinates (t', x', y', z') in B's frame from the assumptions:
(1) that the transformation is linear;
(2) that the spatial coordinates in the directions orthogonal to the direction of motion are unchanged;
(3) that the speed of light c is the same for both A and B, so that $x = t$ in A's frame transforms to $x' = t'$ in B's frame, and likewise $x = -t$ in A's frame transforms to $x' = -t'$ in B's frame;
(4) the definition of speed;
if B is moving at speed v relative to A, then $x = vt$ in A's frame transforms to $x' = 0$ in B's frame;
(5) spatial isotropy; specifically, show that if A thinks B is moving at velocity v, then B must think that A is moving at velocity $-v$, and symmetry (spatial isotropy) between these two situations then fixes the Lorentz γ factor.
Your logic should be precise, and explained in clear, concise English.

2. Scalar product

Suppose that a^m and b^m are two 4-vectors. Show that $a_m b^m$ is a scalar, that is, it is unchanged by any Lorentz transformation. [Hint: For the Minkowski metric of special relativity, $a_m b^m = a^t b^t - a^x b^x - a^y b^y - a^z b^z$. Show that $a'_m b'^m = a_m b^m$. You may assume without proof the familiar result that the 3D scalar product $\boldsymbol{a}.\boldsymbol{b} = a^x b^x + a^y b^y + a^z b^z$ of two 3-vectors is unchanged by any spatial rotation, so it suffices to consider a Lorentz boost, say in the x direction.]

3. The Rules of 4D Perspective

(a) Celestial ellipsoid

What is the photon 4-vector p'^m in a primed frame of reference moving at speed v in the x direction? Argue that the photon 4-vectors in the unprimed and primed frames are related geometrically by the "celestial ellipsoid" transformation illustrated in the notes. Bear in mind that the photon vector is pointed *towards* the observer.

(b) Aberration

The photon 4-vector seen by an observer is the null vector $p^k = E(1, -\boldsymbol{n})$, where E is the photon energy, and $\boldsymbol{n}$ is a unit 3-vector in the direction away from the observer, the minus sign taking into account the fact that the photon vector is pointed towards the observer. An object appears in the unprimed frame at angle θ to the x-direction and in the primed frame at angle θ' to the x-direction. Show that $\mu' \equiv \cos\theta'$ and $\mu \equiv \cos\theta$ are related by

$$\mu' = \frac{\mu + v}{1 + v\mu} \ . \tag{3.1}$$

(c) Redshift

By what factor $a = E'/E$ is the observed photon frequency from the object changed? Express your answer as a function of γ, v, and μ.

(d) Brightness

Photons at frequency E in the unprimed frame appear at frequency E' in the primed frame. Argue that the brightness $F(E)$, the number of photons per log interval of frequency (about E in the unprimed frame, and E' in the primed frame) per unit solid angle per unit time,

$$F(E) \equiv \frac{dN(E)}{d\ln E \, do \, dt} \ . \tag{3.2}$$

goes as

$$\frac{F'(E')}{F(E)} = \frac{E'}{E}\frac{d\mu}{d\mu'} = a^3 \ . \tag{3.3}$$

[Hint: Photons number conservation implies that $dN'(E') = dN(E)$.]

4. Circles on the sky

Show that a circle on the sky Lorentz transforms to a circle on the sky. Let the primed frame be moving at velocity v in the x-direction, let θ be the angle between the x-direction and the direction $\boldsymbol{m}$ to the center of the circle, and let α be the angle between the circle axis $\boldsymbol{m}$ and the photon direction $\boldsymbol{n}$. Show that the angle θ' in the primed frame is given by

$$\tan\theta' = \frac{\sin\theta}{\gamma(v\cos\alpha + \cos\theta)} \ , \tag{4.1}$$

and that the angular radius α' in the primed frame is given by

$$\tan\alpha' = \frac{\sin\alpha}{\gamma(\cos\alpha + v\cos\theta)} \ . \tag{4.2}$$

[Hint: This result was first obtained by Penrose 1958, prior to which it had been widely thought that circles would appear Lorentz-contracted and therefore squashed. The following simple proof was told to me by Engelbert Schucking (NYU). The set of null 4-vectors $p^k = E\{1, -\boldsymbol{n}\}$ on the circle satisfies the Lorentz invariant equation $x_k p^k = 0$, where $x^k = |x|\{-\cos\alpha, \boldsymbol{m}\}$ is a spacelike 4-vector whose spatial components $|x|\boldsymbol{m}$ point to the center of the circle.]