

PHYS 5770 Gravitational Theory Spring 2008. Problem Set 2. Due Tue 12 Feb

1. Equation of motion

Variation of the action yields Lagrange's equations for the motion of a particle in terms of its Lagrangian $\mathcal{L}(x^\mu, u^\mu)$

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial u^\lambda} = \frac{\partial \mathcal{L}}{\partial x^\lambda} \quad (1.1)$$

where τ is the particle's proper time, and u^μ its 4-velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau} . \quad (1.2)$$

The coordinates x^μ and velocities u^μ in this approach are taken to be independent coordinates, so the velocity partial derivatives $\partial/\partial u^\lambda$ in equation (1.1) are to be interpreted as being done with the coordinates x^μ held fixed, and conversely the coordinate partial derivatives $\partial/\partial x^\lambda$ are done with the velocities u^μ held fixed. In general relativity, the effective Lagrangian of a free particle can be taken to be

$$\mathcal{L} = g_{\mu\nu}(x) u^\mu u^\nu \quad (1.3)$$

with the metric $g_{\mu\nu}$ being a function only of the coordinates x^μ , not the velocities u^μ . Show that Lagrange's equations (1.1) are equivalent to the usual equations of motion

$$\frac{d\mathbf{u}}{d\tau} = 0 . \quad (1.4)$$

[Hint: First argue that $\mathbf{g}^\lambda \cdot \mathbf{g}_\mu = \delta_\mu^\lambda$. Then from $\partial(\mathbf{g}^\lambda \cdot \mathbf{g}_\mu)/\partial x^\nu = 0$ and from the definition of the connection coefficients, $\partial \mathbf{g}_\mu/\partial x^\nu \equiv \Gamma_{\mu\nu}^\lambda \mathbf{g}_\lambda$, deduce that $\partial \mathbf{g}^\lambda/\partial x^\nu = -\Gamma_{\mu\nu}^\lambda \mathbf{g}^\mu$. Show that the equation of motion (1.4), with $\mathbf{u} = \mathbf{g}^\lambda u_\lambda$, becomes $\mathbf{g}^\lambda (du_\lambda/d\tau - \Gamma_{\mu\lambda\nu} u^\mu u^\nu) = 0$, where $\Gamma_{\mu\lambda\nu} \equiv g_{\mu\kappa} \Gamma_{\lambda\nu}^\kappa$. Now argue that $\partial \mathcal{L}/\partial u^\lambda = 2u_\lambda$, and that $\partial \mathcal{L}/\partial x^\lambda = 2\Gamma_{\mu\nu\lambda} u^\mu u^\nu$. To complete, you'll need to invoke the no-torsion symmetry $\Gamma_{\mu\nu\lambda} = \Gamma_{\mu\lambda\nu}$.]

2. Constants of motion

Suppose that thanks to some symmetry the metric is independent of some coordinate, say ϕ . Argue from the Lagrange's equations (1.1) that the corresponding covariant (index down) component of the 4-velocity is a constant of motion

$$u_\phi = \text{constant} . \quad (2.1)$$

If integrals of motion exist, this is by far the simplest way to discover them. [Hint: This is a short problem.]

3. Earth metric

The metric just above the surface of the Earth is well-approximated by

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.1)$$

where

$$\Phi(r) = -\frac{GM}{r} \quad (3.2)$$

is the familiar Newtonian gravitational potential.

(a) Proper time

Consider an object at fixed radius r , moving along the equator $\theta = \pi/2$ with constant non-relativistic velocity $r d\phi/dt = v$. Compare the proper time of this object with that at rest at infinity. [Hint: Work to first order in the potential Φ . Regard v^2 as first order in Φ .]

(b) Orbits

Consider a satellite in orbit about the Earth. The conservation of energy E per unit mass, angular momentum L per unit mass, and rest mass per unit mass are expressed by

$$u_t = E, \quad u_\phi = -L, \quad u_\mu u^\mu = 1. \quad (3.3)$$

For equatorial orbits, $\theta = \pi/2$, show that the radial component u^r of the 4-velocity satisfies

$$u^r = \sqrt{2(\Delta E - U)} \quad (3.4)$$

where ΔE is the energy per unit mass of the particle excluding its rest mass energy

$$\Delta E = E - 1 \quad (3.5)$$

and the effective potential U is

$$U = \Phi + \frac{L^2}{2r^2}. \quad (3.6)$$

[Hint: Neglect air resistance. Remember to work to first order in Φ . Are ΔE and L^2 first order in Φ ? Why?]

(c) Circular orbits

From the condition that the potential U be an extremum, find the circular orbital velocity $v = r d\phi/dt$ of a satellite at radius r .

(d) Special versus general relativistic corrections

Compare the proper time of a satellite to that of a person on the Earth. Which correction is the largest, and what are their signs: the special relativistic correction arising from the orbital velocity of the satellite, or the general relativistic correction from the gravitational potential? Consider satellites (i) in low Earth orbit, such as the International Space Station, (ii) in medium Earth orbit, such as a GPS satellite, (iii) in geostationary orbit. [Hint: Google the numbers that you may need.]

4. Rindler space

(a) Rindler coordinates

Let $\{t, x, y, z\}$ be globally inertial coordinates of Minkowski space, with metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 . \quad (4.1)$$

Define Rindler coordinates $\{\alpha, l, y, z\}$ by

$$t = l \sinh \alpha , \quad x = l \cosh \alpha . \quad (4.2)$$

What do the Rindler coordinates signify physically? [Hint: What do l and α mean? What do lines of constant l signify? What do lines of constant α signify?]

(b) Rindler metric

Show that the Rindler metric is

$$ds^2 = l^2 d\alpha^2 - dl^2 - dy^2 - dz^2 . \quad (4.3)$$

(c) Constants of motion

For geodesic motion, the covariant components u_α , u_y , and u_z of the 4-velocity are constants of motion. What do these constants imply for behavior of the 4-velocity u^κ in geodesic motion? Interpret your results physically.

(d) Connections

Compute the connection coefficients $\Gamma_{\mu\nu}^\kappa$ of the Rindler metric.

(e) Riemann tensor

Compute the Riemann tensor $R_{\kappa\lambda\mu\nu}$. All the components should be zero. Why?

(f) 4-velocity

Argue that the 4-velocity $u^\kappa \equiv dx^\kappa/d\tau$ of comoving observers in Rindler space (observers following worldlines at fixed l , y , and z) is

$$u^\kappa = \{1/l, 0, 0, 0\} . \quad (4.4)$$

(g) 4-acceleration

Compute the 4-acceleration a^κ of comoving observers from the geodesic equation

$$\frac{du^\kappa}{d\tau} + \Gamma_{\mu\nu}^\kappa u^\mu u^\nu = a^\kappa . \quad (4.5)$$

(h) Proper acceleration

What is the proper acceleration a of a comoving observer?