PHYS 5770 Gravitational Theory Spring 2008. Problem Set 3. Due Tue 26 Feb

1. Equations of motion in weak gravity

Consider the Newtonian metric

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \tag{1.1}$$

where $\Phi(x, y, z)$ is the familiar Newtonian gravitational potential, a function only of the spatial coordinates x, y, z, not of time t.

(a) Connection coefficients

Confirm that the non-zero connection coefficients are (coefficients as below but with the last two indices swapped are the same by the no-torsion condition $\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$)

$$\Gamma^t_{ti} = \Gamma^i_{tt} = \Gamma^i_{jj} = -\Gamma^j_{ji} = -\Gamma^i_{ii} = \frac{\partial\Phi}{\partial x^i} \quad (i \neq j = x, y, z) \ . \tag{1.2}$$

[Hint: Work to linear order in Φ. You are welcome to use the mathematica notebook metric.nb posted on the website, but if you do, please tell me.]

(b) Energy of a massive particle

Consider a massive, non-relativistic particle moving with 4-velocity $u^\mu \equiv dx^\mu/d\tau = \{u^t, u^x, u^y, u^z\}$. Show that $u_\mu u^\mu = 1$ implies that

$$u^t = 1 + \frac{1}{2}u^2 - \Phi \tag{1.3}$$

whereas

$$u_t = 1 + \frac{1}{2}u^2 + \Phi \tag{1.4}$$

where $u \equiv \left[(u^x)^2 + (u^y)^2 + (u^z)^2\right]^{1/2}$. One of u^t or u_t is constant. Which one? [Hint: Work to linear order in Φ. Note that u^2 is of linear order in Φ. As regards which of u^t or u_t is constant, notice that the metric is independent of time because $\Phi(x, y, z)$ is being assumed to be a function only of the spatial coordinates x, y, z, not of time t. You are welcome to quote the results of Problem Set 2.]

(c) Equation of motion of a massive particle

From the geodesic equation

$$\frac{du^\kappa}{d\tau} + \Gamma^\kappa_{\mu\nu} u^\mu u^\nu = 0 \tag{1.5}$$

show that

$$\frac{du^i}{dt} = -\frac{\partial\Phi}{\partial x^i} \quad i = x, y, z \ . \tag{1.6}$$

Why is it legitimate to replace $d\tau$ by dt? Show further that

$$\frac{du^t}{dt} = -2\, u^i \frac{\partial\Phi}{\partial x^i} \tag{1.7}$$

with implicit summation over $i = x, y, z$. Does the result agree with what you'd expect from equation (1.3)? [Hint: A consistent perturbative approach is to keep only the lowest order non-vanishing parts of an equation, discarding the higher order parts as negligible.]

(d) Energy of a massless particle

For a massless particle, the proper time along a geodesic is zero, and the affine parameter λ must be used instead of the proper time. The 4-velocity of a massless particle can be defined to be (and really this is just the 4-momentum up to an arbitrary overall factor) $v^\mu \equiv dx^\mu/d\lambda = \{v^t, v^x, v^y, v^z\}$. Show that $v_\mu v^\mu = 0$ implies that

$$v^t = (1 - 2\Phi)v \tag{1.8}$$

whereas

$$v_t = v \tag{1.9}$$

where $v \equiv \left[(v^x)^2 + (v^y)^2 + (v^z)^2\right]^{1/2}$. One of v^t or v_t is constant. Which one?

(e) Equation of motion of a massless particle

From the geodesic equation

$$\frac{dv^\kappa}{d\lambda} + \Gamma^\kappa_{\mu\nu} v^\mu v^\nu = 0 \tag{1.10}$$

show that the spatial components $\boldsymbol{v} \equiv \{v^x, v^y, v^z\}$ satisfy

$$\frac{d\boldsymbol{v}}{d\lambda} = 2\,\boldsymbol{v} \times \left(\boldsymbol{v} \times \frac{\partial \Phi}{\partial \boldsymbol{x}}\right) \tag{1.11}$$

where boldface symbols represent 3D vectors, and in particular $\partial\Phi/\partial\boldsymbol{x}$ is the spatial 3D gradient $\partial\Phi/\partial\boldsymbol{x} \equiv \partial\Phi/\partial x^i = \{\partial\Phi/\partial x, \partial\Phi/\partial y, \partial\Phi/\partial z\}$. [Hint: Recall that the 3D vector triple product satisfies $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.]

(f) Interpret

Interpret your answer, equation (1.11). In what ways does this equation for the acceleration of photons differ from the equation governing the acceleration of massive particles? [Hint: Without loss of generality, the affine parameter can be normalized so that the photon speed is one, $v = 1$, so that $\boldsymbol{v}$ is a unit vector representing the direction of the photon.]

(g) Redshift

Consider an observer who happens to be at rest in the Newtonian metric, so that $u^x = u^y = u^z = 0$. Argue that the energy of a photon observed by this observer, relative to an observer at rest at zero potential, is

$$u^\mu v_\mu = 1 - \Phi\ . \tag{1.12}$$

Does the observed photon have higher or lower energy in a deeper potential well?

2. Geodesics in the Schwarzschild metric

The Schwarzschild metric is

$$ds^2 = B(r)\, dt^2 - \frac{1}{B(r)}\, dr^2 - r^2\, (d\theta^2 + \sin^2\!\theta\, d\phi^2) \tag{2.1}$$

where

$$B(r) = 1 - \frac{2M}{r}\ . \tag{2.2}$$

Without loss of generality, the trajectory of a freely falling particle may be taken to lie in the equatorial plane, $\theta = \pi/2$. For a massive particle, conservation of energy per unit rest mass E, angular momentum per unit rest mass L, and rest mass per unit rest mass implies that the 4-velocity $u^\mu \equiv dx^\mu/d\tau$ satisfies

$$\begin{aligned} u_t &= E\ , \\ u_\phi &= -L\ , \\ u_\mu u^\mu &= 1\ . \end{aligned} \tag{2.3}$$

(a) Effective potential

Show that the radial component u^r of the 4-velocity satisfies

$$u^r = \pm\, \left(E^2 - U\right)^{1/2} \tag{2.4}$$

where U is the effective potential

$$U = B\left(1 + \frac{L^2}{r^2}\right)\ . \tag{2.5}$$

(b) Radial free-fall

What is the proper time τ for an observer to free-fall from radius r to the singularity at zero radius, for the particular case of an observer who falls radially from rest at infinity. [Hint: What are the energy E and angular momentum L for an observer who falls radially starting from rest at infinity?]

(c) Circular orbits

Circular orbits occur where the effective potential U is an extremum. Find the radii at which this occurs, as a function of angular momentum L. You should find that solutions exist only if the absolute value $|L|$ of the angular momentum exceeds a certain critical value. What is this critical value?

(d) Range of orbits

Identify the ranges of radii over which circular orbits are: (i) stable, (ii) unstable, (iii) non-existent. [Hint: Stability depends on whether the extremum of the effective potential is a minimum or a maximum. Which is which?]

(e) Angular momentum and energy in circular orbit

Show that the angular momentum per unit mass for a circular orbit at radius r satisfies

$$|L| = \frac{r}{(r/M - 3)^{1/2}} \ , \tag{2.6}$$

and hence show also that the energy per unit mass in the circular orbit is

$$E = \frac{r - 2M}{[r(r - 3M)]^{1/2}} \ . \tag{2.7}$$

(f) Drop in orbit

There is a certain circular orbit that has the same energy as a massive particle at rest at infinity. This is useful for starship captains to know, because it is possible to drop into this orbit using only a small amount of energy. What is the radius of the orbit? Is it stable or unstable? [Hint: What is the energy E of a particle at rest at infinity?]

(g) Photon sphere

There is a radius where photons can orbit in circular orbits. What is the radius of this orbit? [Hint: Photons can be taken as the limit of a massive particle whose energy per unit mass E vastly exceeds its rest mass energy per unit mass, which is 1.]

(h) Orbital period

Show that the orbital period t, as measured by an observer at rest at infinity, of a particle in circular orbit at radius r is given by Kepler's 3rd law (remarkably, Kepler's 3rd law remains true even in the fully general relativistic case, as long as t is taken to be the time measured at infinity)

$$\frac{GMt^2}{(2\pi)^2} = r^3 \ . \tag{2.8}$$

[Hint: Argue that the azimuthal angle ϕ evolves according to $d\phi/dt = u^\phi/u^t = LB/(Er^2)$.]