

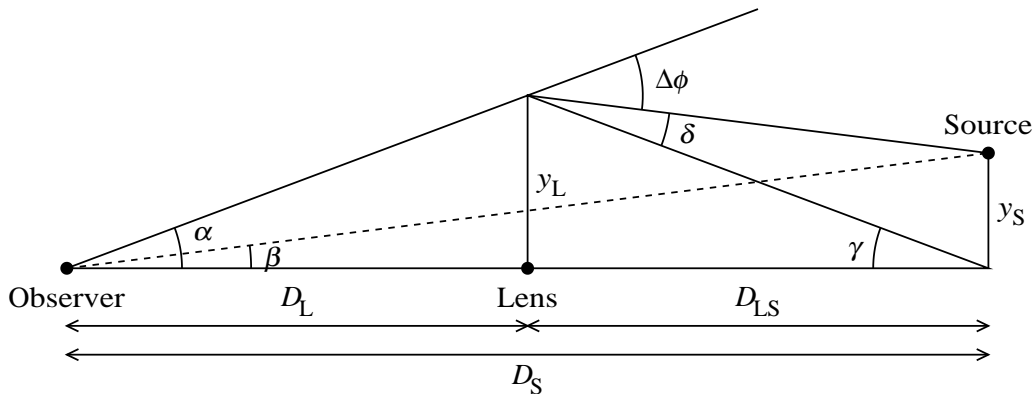
PHYS 5770 Grav Theory Spring 2008. Problem Set 4. Due Tue 18 Mar

1. Gravitational lensing

In a previous problem you found that, in the weak field limit, light passing a spherical mass M at impact parameter y is deflected by angle

$$\Delta\phi = \frac{4GM}{yc^2} . \quad (1.1)$$

(a) Lensing equation



Argue that the deflection angle $\Delta\phi$ is related to the angles illustrated in the lensing diagram above by

$$\Delta\phi = \alpha + \gamma - \delta . \quad (1.2)$$

Hence obtain the “lensing equation” in the form commonly used by astronomers

$$\beta = \alpha - \frac{\alpha_E^2}{\alpha} \quad (1.3)$$

where

$$\alpha_E = \left(\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S} \right)^{1/2} . \quad (1.4)$$

(b) Einstein ring around the Sun?

The case $\alpha = \alpha_E$ evidently corresponds to the case where the source is exactly behind the lens, $\beta = 0$. In this case the lensed source appears as an “Einstein ring” of light around the lens. Could there be an Einstein ring around the Sun, as seen from Earth?

(c) Einstein ring around Sgr A*

What is the maximum possible angular size of an Einstein ring around the $4 \times 10^6 M_\odot$ black hole at the center of our Milky Way, 8 kpc away?

2. Geodesics in the FRW geometry

The Friedmann-Robertson-Walker metric of cosmology is

$$ds^2 = dt^2 - a(t)^2 \left[dx_{\parallel}^2 + \frac{\sin^2(\kappa^{1/2} x_{\parallel})}{\kappa} (d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (2.1)$$

where κ is a constant, the curvature constant. Note that equation (2.1) is valid for all values of κ , including zero and negative values: there is no need to consider the cases separately.

(a) Conservation of generalized momentum

Consider a particle moving along a geodesic in the radial direction, so that $d\theta = d\phi = 0$. Argue that the Lagrangian equations of motion

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial u^{x_{\parallel}}} = \frac{\partial \mathcal{L}}{\partial x_{\parallel}} \quad (2.2)$$

with effective Lagrangian

$$\mathcal{L} = g_{\mu\nu} u^{\mu} u^{\nu} \quad (2.3)$$

imply that

$$u_{x_{\parallel}} = \text{constant} . \quad (2.4)$$

Argue further from the same Lagrangian equations of motion that the assumption of a radial geodesic is valid because

$$u_{\theta} = u_{\phi} = 0 \quad (2.5)$$

is a consistent solution. [Hint: Unlike in previous problem sets, the metric $g_{\mu\nu}$ now depends on the coordinate x_{\parallel} . But for radial geodesics with $u^{\theta} = u^{\phi} = 0$, the possible contributions from derivatives of the metric vanish.]

(b) Proper momentum

Argue that a proper interval of distance along the radial geodesic is adx_{\parallel} . Hence show from equation (2.4) that the proper momentum p_{\parallel} of the particle relative to comoving observers (who are at rest in the FRW metric) evolves as

$$p_{\parallel} \equiv ma \frac{dx_{\parallel}}{d\tau} \propto \frac{1}{a} . \quad (2.6)$$

(c) Redshift

What relation does your result (2.6) imply between the redshift $1 + z$ of a distant object observed on Earth and the expansion factor a since the object emitted its light? [Hint: Equation (2.6) is valid for massless as well as massive particles. Why?]

(d) Temperature of the CMB

Argue from the above results that the temperature T of the CMB evolves with cosmic scale factor as

$$T \propto \frac{1}{a} . \quad (2.7)$$

3. Mass-energy in an FRW Universe

(a) First law

The first law of thermodynamics for adiabatic expansion is built into Friedmann's equations (= Einstein's equations for the FRW metric):

$$d(\rho a^3) + p da^3 = 0 . \quad (3.1)$$

How does the density ρ evolve with cosmic scale factor for a species with equation of state $p/\rho = w$? You should get an answer of the form

$$\rho \propto a^n . \quad (3.2)$$

(b) Radiation

Given that blackbody radiation has an energy density $\rho \propto T^4$, is your result (3.2) consistent with the result (2.7) of the previous question?

(c) Attractive or repulsive?

For what equation of state w is the mass-energy attractive or repulsive? Consider in particular the cases of "matter", "radiation", and "vacuum" energy.