PHYS 5770 Grav Theory Spring 2008. Problem Set 5. Due Tue 15 Apr

1. Tidal forces falling into a Schwarzschild black hole

In the Gullstrand-Painlevé tetrad, the non-zero components of the tetrad-frame Riemann tensor are

$$-\tfrac{1}{2}R_{trtr} = R_{t\theta t\theta} = R_{t\phi t\phi} = -R_{r\theta r\theta} = -R_{r\phi r\phi} = \tfrac{1}{2}R_{\theta\phi\theta\phi} = C \tag{1.1}$$

where

$$C = -\frac{M}{r^3} \tag{1.2}$$

is the Weyl scalar.

(a) Tidal forces

By construction of the Gullstrand-Painlevé tetrad, a person who falls radially from zero velocity at infinity is at rest in the Gullstrand-Painlevé tetrad, with tetrad-frame 4-velocity $u^m = \{1, 0, 0, 0\}$. From the equation of geodesic deviation

$$\frac{D^2 \delta\xi_m}{D\tau^2} + R_{klmn}\delta\xi^k u^l u^n = 0 \tag{1.3}$$

deduce the tidal acceleration on the person in the radial and transverse directions. Does the tidal acceleration stretch or compress? [Hint: The equation of geodesic deviation gives the proper acceleration between two points a small distance $\delta\xi^m$ apart, where ξ^m are the locally inertial coordinates of the tetrad frame. Notice that this problem is much easier to solve with tetrads than with the traditional coordinate approach.]

(b) Choose a black hole to fall into

What is the mass of the black hole for which the tidal acceleration M/r^3 is 1 gee per meter at the horizon? If you wanted to fall through the horizon of a black hole without first being torn apart, what mass of black hole would you choose? [Hint: 1 gee is the gravitational acceleration at the surface of the Earth.]

(c) Time to die

In problem set 3 you showed that the proper time to free-fall radially from radius r to the singularity of a Schwarzschild black hole is

$$\tau = \frac{\sqrt{2}}{3}\sqrt{\frac{r^3}{M}}\ . \tag{1.4}$$

How long, in seconds, does it take to fall to the singularity from the place where the tidal acceleration is 1 gee per meter?

2. Constant density star

A variation of the following calculation (done in a very different way) of a static spherically symmetric gravitating system convinced Einstein (1939, Ann. Math. 40, 922) that black holes cannot exist.

In a spherically symmetric static spacetime, Einstein's equations reduce to an equation for the mass M interior to r

$$\frac{dM}{dr} = 4\pi r^2 \rho \ , \tag{2.1}$$

and to the Volkov-Oppenheimer equation of hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{(\rho + p)(M + 4\pi r^3 p)}{r^2(1 - 2M/r)} \ . \tag{2.2}$$

(a) Interior mass

Suppose that the density ρ is constant. From equation (2.1) obtain an expression for the interior mass M as a function of radius r and the density ρ. [Hint: This is easy.]

(b) Hydrostatic equilibrium

Given your expression for M, show that the Volkov-Oppenheimer equation rearranges to

$$\int_{p_c} \frac{dp}{(\rho + p)(\rho + 3p)} = -\int_0 \frac{4\pi r \, dr}{3 - 8\pi r^2 \rho} \tag{2.3}$$

where p_c is the central pressure, where the radius is zero, $r = 0$.

(c) Solve

Integrate equation (2.3). From the integral evaluated at the edge of the star, where the pressure is zero, $p = 0$, and the radius is the stellar radius, $r = R_\star$, argue that

$$\frac{\rho + 3p_c}{\rho + p_c} = \sqrt{\frac{1}{1 - 2M_\star/R_\star}} \tag{2.4}$$

where $M_\star \equiv \frac{4}{3}\pi\rho R_\star^3$ is the total mass of the star.

(d) Limits

From the condition that the central pressure be positive and finite, $0 < p_c < \infty$, deduce that

$$0 < \frac{2M_\star}{R_\star} < \frac{8}{9} \ . \tag{2.5}$$

(e) Comment

Comment on what equation (2.5) implies physically. [Hint: What is the Schwarzschild radius?]