PHYS 5770 Grav Theory Spring 2008. Problem Set 6. Due Tue 29 Apr

1. Will you be torn apart when two black holes merge?

This question is posed on behalf of Phil Plait, the Bad Astronomer, whose upcoming (October 2008) book "Death from the Skies!" contains a chapter "Seven ways a black hole can kill you". One of the ways, says Phil, is to stand near a pair of merging black holes, and be torn apart by the tidal forces from the gravitational waves. Is it true?

(a) Quadrupole moment

Consider a pair of masses M_1 and M_2 in circular orbit, with position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ relative to their center of mass. Argue that the quadrupole moment $\not{I}_{ij}$ of the mass distribution

$$\not{I}_{ij} \equiv \sum_{\text{masses } a} M_a (r_{a,i}\, r_{a,j} - \tfrac{1}{3}\, \delta_{ij}\, r_a^2) \tag{1.1}$$

is

$$\not{I}_{ij} = m r^2 (\hat{r}_i \hat{r}_j - \tfrac{1}{3} \delta_{ij}) \tag{1.2}$$

where $\boldsymbol{r} \equiv \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the orbital separation, and m is the reduced mass

$$m \equiv \frac{M_1 M_2}{M} \ , \quad M \equiv M_1 + M_2 \ . \tag{1.3}$$

[Hint: Assume for simplicity that the orbit is described by classical Newtonian mechanics. After all, our aim is to decide whether we die, and Newton is good enough for that.]

(b) Tensor components

Suppose that the orbital plane is inclined at inclination angle ι to the line-of-sight. Choose the observer's locally inertial frame so that the x-axis is the line-of-sight direction from the center of mass of the binary to the observer, and the y-axis points in the plane of the orbit. Argue that the orbital separation $\boldsymbol{r}$ is

$$\boldsymbol{r} = r \left[(\hat{x} \cos\iota - \hat{z} \sin\iota) \cos\omega t + \hat{y} \sin\omega t \right] \tag{1.4}$$

where ω is the orbital frequency (units $G = 1$)

$$\omega^2 = \frac{M}{r^3} \ . \tag{1.5}$$

Deduce that the tensor components of the quadrupole moment are

$$\not{I}_+ \equiv \tfrac{1}{2}(\not{I}_{yy} - \not{I}_{zz}) = \tfrac{1}{4} m r^2 \left[\cos^2\iota - (1 + \sin^2\iota) \cos 2\omega t \right] \ , \tag{1.6a}$$

$$\not{I}_\times \equiv \not{I}_{yz} = -\tfrac{1}{2} m r^2 \sin\iota \sin 2\omega t \ . \tag{1.6b}$$

(c) Tensor perturbation

Deduce the tensor perturbations h_+ and $h_\times$ at distance x from the orbiting masses from the quadrupole formula

$$h_{ij} = -\frac{1}{x} \ddot{\not{I}}_{ij \, \text{tensor}} \ . \tag{1.7}$$

(d) Tidal forces

For a gravitational wave propagating in the x-direction in empty space, the non-zero components of the Riemann tensor of the perturbed Minkowski space are

$$-R_{tyty} = R_{tztz} = R_{tyxy} = -R_{tzxz} = -R_{xyxy} = R_{xzxz} = \ddot{h}_+ \ , \tag{1.8a}$$

$$-R_{tytz} = R_{tyxz} = R_{tzxy} = -R_{xyxz} = \ddot{h}_\times \ . \tag{1.8b}$$

From the equation of geodesic deviation

$$\frac{D^2 \delta\xi_m}{D\tau^2} + R_{klmn} \delta\xi^k u^l u^n = 0 \tag{1.9}$$

deduce the tidal forces on a person moving non-relativistically. [Hint: If a person is moving non-relativistically, it is legitimate to take the person's 4-velocity to be $u^m = \{1, 0, 0, 0\}$. Why?]

(e) Comment

What is your advice to Phil Plait? [Hint: What you need here is rough estimates. Consider both supermassive and stellar-sized black holes. To make things sensible, you should require that you, the observer, be (a) outside the horizon, and (b) outside the point at which the static tidal force of the black hole would tear you apart even without gravitational waves. You may find it convenient to define the mass M_g of a black hole whose tidal force at the horizon is 1 gee per meter

$$g = \frac{1}{M_g^2} \tag{1.10}$$

which you figured out in a previous Problem Set.]

2. Lense-Thirring precession

(a) Geodesic equation

In perturbed Minkowski space, and in Newtonian gauge, the non-vanishing tetrad connections that depend on the vector perturbation $\boldsymbol{W} \equiv W_i$ are

$$\Gamma_{tij} = \tfrac{1}{2}(\partial_i W_j + \partial_j W_i) \ , \quad \Gamma_{ijt} = -\tfrac{1}{2}(\partial_i W_j - \partial_j W_i) \ . \tag{2.1}$$

From the equation of motion

$$\frac{dp^k}{d\tau} + \Gamma^k_{mn} p^m u^n = 0 \tag{2.2}$$

show that, for an object that is moving at non-relativistic speed, the time component p^t of a 4-vector p^k is constant to linear order, while the spatial components $\boldsymbol{p}$ evolve as

$$\frac{d\boldsymbol{p}}{d\tau} = -\frac{1}{2}\boldsymbol{p} \times (\boldsymbol{\partial} \times \boldsymbol{W}) \ . \tag{2.3}$$

[Hint: Recall the 3-vector formula $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$. If the object is non-relativistic, then it is legitimate to take the 4-velocity of the object to be $u^m = \{1, 0, 0, 0\}$. Why?]

(b) Angular velocity

As shown in the notes, a body of angular momentum $\boldsymbol{L}$ produces a vector perturbation $\boldsymbol{W}$ at spatial position $\boldsymbol{x}$ of

$$\boldsymbol{W}(\boldsymbol{x}) = -\frac{2}{x^2}\,\hat{\boldsymbol{x}} \times \boldsymbol{L} \ . \tag{2.4}$$

From this and equation (2.3), deduce the angular velocity of precession induced by such a body. This is Lense-Thirring precession.

(c) Earth

Estimate the angular velocity of the Lense-Thirring precession of a gyroscope in orbit around the Earth. Express your answer in arcseconds per year.

(d) Quadrupole precession

There is also a purely Newtonian precession that is produced by plain old Newtonian gravity on an object with a quadrupole moment. If you wanted to test the Lense-Thirring effect with a gyroscope in orbit around the Earth, what would you do to avoid contamination by Newtonian quadrupole precession?