1 Special Relativity

1.1 Concept Questions

1. What does $c =$ universal constant mean? What is speed? What is distance? What is time?

2. $c + c = c$. How can that be possible?

3. The first postulate of SR asserts that spacetime forms a 4-dimensional continuum. The fourth postulate of SR asserts that spacetime has no absolute existence. Isn't that a contradiction?

4. How can two people moving relative to each other at near c both think each other's clock runs slow?

5. How can two people moving relative to each other at near c both think the other is Lorentz-contracted?

6. All paradoxes in SR have the same solution. In one word, what is that solution?

7. All conceptual paradoxes in SR can be understood by drawing what kind of diagram?

8. Your twin takes a trip to α Cen at near c, then returns to Earth at near c. Meeting your twin, you see that the twin has aged less than you. But from your twin's perspective, it was you that receded at near c, then returned at near c, so your twin thinks you aged less. Is it true?

9. Blobs in the jet of the galaxy M87 have been tracked by the Hubble Space Telescope to be moving at almost $6c$. Does this violate SR?

10. If you watch an object move at near c, does it actually appear Lorentz-contracted? Explain.

11. You speed towards the center of our Galaxy, the Milky Way, at near c. Does the center appear to you closer or farther away?

12. You go on a trip to the center of the Milky Way, 30,000 lightyears distant, at near c. How long does the trip take you?

13. You surf a light ray from a distant quasar to Earth. How much time does the trip take, from your perspective?

14. If light is a wave, what is waving?

15. As you surf the light ray, how fast does it appear to vibrate?

16. How does the phase of a light ray vary along the light ray? Draw surfaces of constant phase on a spacetime diagram.

17. You see a distant galaxy at a redshift of $z = 1$. If you could see a clock on the galaxy, how fast would the clock appear to tick? Could this be tested observationally?

18. You take a trip to α Cen at near c, then instantaneously accelerate to return at near c. If you are looking through a telescope at a clock on the Earth while you instantaneously accelerate, what do you see happen to the clock?

19. In what sense is time an imaginary spatial dimension?

20. In what sense is a Lorentz boost a rotation by an imaginary angle?

21. You know what it means for an object to be rotating at constant angular velocity. What does it mean for an object to be boosting at a constant rate?

22. A wheel is spinning so that its rim is moving at near c. The rim is Lorentz-contracted, but the spokes are not. How can that be?

23. You watch a wheel rotate at near the speed of light. The spokes appear bent. How can that be?

24. Energy and momentum are unified in SR. Explain.

25. In what sense is mass equivalent to energy in SR? In what sense is mass different from energy?

26. Why is the Minkowski metric unchanged by a Lorentz transformation?

27. What is the best way to program Lorentz transformations on a computer?

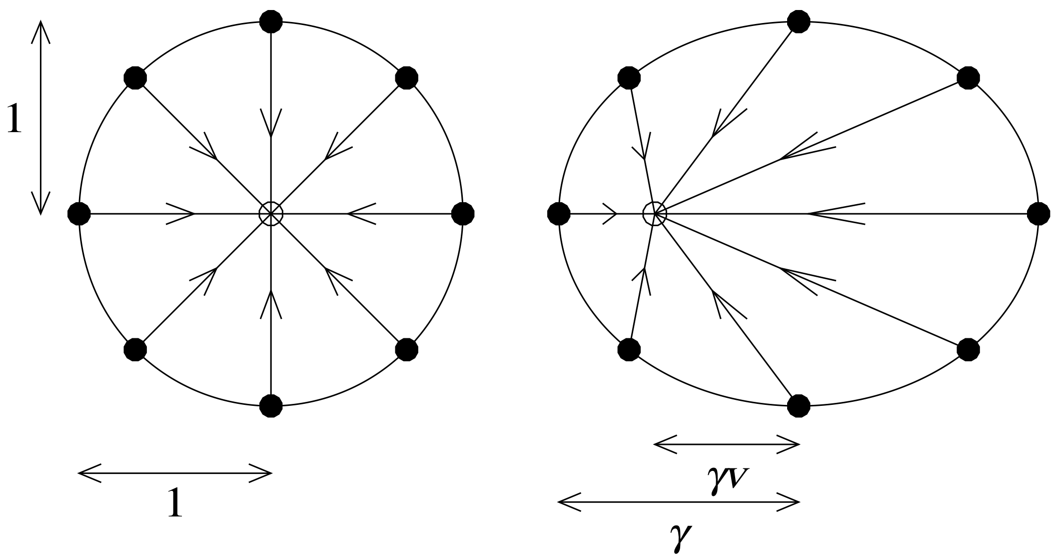
1.2 What's important in SR

See
`http://casa.colorado.edu/~ajsh/sr/`

1. Postulates of SR.
2. Understanding conceptually the unification of space and time implied by SR.
   (a) Spacetime diagrams.
   (b) Simultaneity.
   (c) Understanding the paradoxes of relativity – time dilation, Lorentz contraction, the twin paradox.
3. The mathematics of spacetime transformations – Lorentz transformations.
   (a) 4-vectors.
   (b) Energy-momentum 4-vector. $E = mc^2$.
   (c) The energy-momentum 4-vector of massless particles, such as photons.
   (d) Invariant spacetime distance.
   (e) Minkowski metric.
4. Phenomenology – The Rules of 4D Perspective.

1.3 The rules of 4-dimensional perspective

The diagram below illustrates the rules of 4-dimensional perspective, also called "special relativistic beaming", which describe how a scene appears when you move through it at near light speed.



On the left, you are at rest relative to the scene. Imagine painting the scene on a celestial sphere around you. The arrows represent the directions of light rays (photons) from the scene on the celestial sphere to you at the center.

On the right, you are moving to the right through the scene, at some fraction of the speed of light. The celestial sphere is stretched along the direction of your motion into a celestial ellipsoid. You, the observer, are not at the center of the ellipsoid, but rather at one of its foci (the left one, if you are moving to the right). The scene appears relativistically aberrated, which is to say concentrated ahead of you, and expanded behind you.

The lengths of the arrows are proportional to the energies, or frequencies, of the photons that you see. When you are moving through the scene at near light speed, the arrows ahead of you, in your direction of motion, are longer than at rest, so you see the photons blue-shifted, increased in energy, increased in frequency. Conversely, the arrows behind you are shorter than at rest, so you see the photons red-shifted, decreased in energy, decreased in frequency. Since photons are good clocks, the change in photon frequency also tells you how fast or slow clocks attached to the scene appear to you to run.

Numbers? On the right, you are moving through the scene at $v = 0.6\,c$. The celestial ellipsoid is stretched along the direction of your motion by the Lorentz gamma factor, which here is $\gamma = 1/\sqrt{1 - 0.6^2} = 1.25$. The focus of the celestial ellipsoid, where you the observer are, is displaced from center by $\gamma v = 1.25 \times 0.6 = 0.75$.

1.4 Scalar spacetime interval

The quantity

$$
\begin{aligned}
\Delta s^2 &= \Delta t^2 - \Delta r^2 \\
&= \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2
\end{aligned}
\tag{1}
$$

remains unchanged under a Lorentz transformation. Such quantities are called **scalars**. Lorentz transformations can be defined as linear spacetime transformations that leave Δs^2 invariant.

The single scalar spacetime squared interval Δs^2 replaces the two scalar quantities

$$
\begin{array}{rl}
\text{time interval} & \Delta t \\
\text{distance interval} & \Delta r = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}
\end{array}
\tag{2}
$$

of classical Galilean spacetime.

Note. Some authors (including me!) prefer the convention

$$
\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2
\tag{3}
$$

which is minus the above.

1.5 4-vectors

A **4-vector** in SR is a quantity $a^m = \{a^t, a^x, a^y, a^z\}$ that transforms under Lorentz transformations like an interval $\{t, x, y, z\}$ of spacetime

$$
a'^m = L^m{}_n a^n
\tag{4}
$$

where $L^m{}_n$ denotes a Lorentz transformation.

In the case that the Lorentz transformation is a Lorentz boost along the x-axis, the transformation is

$$
\begin{pmatrix} a'^t \\ a'^x \\ a'^y \\ a'^z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^t \\ a^x \\ a^y \\ a^z \end{pmatrix}
\tag{5}
$$

with inverse

$$
\begin{pmatrix} a^t \\ a^x \\ a^y \\ a^z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a'^t \\ a'^x \\ a'^y \\ a'^z \end{pmatrix} .
\tag{6}
$$

1.6 Index notation

In special and general relativity it is convenient to introduce 2 versions of the same 4-vector quantity, one with raised indices, called the **contravariant** components of the 4-vector,

$$a^m \equiv \{a^t, a^x, a^y, a^z\} \ , \tag{7}$$

and one with lowered indices called the **covariant** components of the 4-vector,

$$a_m \equiv \{a^t, -a^x, -a^y, -a^z\} \ . \tag{8}$$

(The naming is crazy, and you do not need to remember it.)

The indices run over $m = t, x, y, z$, or sometimes $m = 0, 1, 2, 3$.

Why introduce raised and lowered indices? Because

$$\begin{aligned} a_m a^m \equiv \sum_m a_m a^m &= a_t a^t + a_x a^x + a_y a^y + a_z a^z \\ &= (a^t)^2 - (a^x)^2 - (a^y)^2 - (a^z)^2 \end{aligned} \tag{9}$$

is a Lorentz scalar.

SR and GR widely adopt the **implicit summation convention**, according to which paired indices are explicitly summed over. Invariably, one of a pair of repeated indices is raised, the other lowered.

1.7 Minkowski metric

The scalar spacetime squared interval Δs^2 associated with an interval $\Delta x^m = \{\Delta t, \Delta \boldsymbol{r}\} = \{\Delta t, \Delta x, \Delta y, \Delta z\}$ can be written

$$\begin{aligned} \Delta s^2 &= \Delta x_m \Delta x^m \\ &= \eta_{mn} \Delta x^m \Delta x^n \end{aligned} \tag{10}$$

where η_{mn} is the **Minkowski** metric

$$\eta_{mn} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{11}$$

1.8 Timelike, lightlike, spacelike

A spacetime interval Δx^m is called

$$\begin{array}{cl} \text{timelike} & \text{if } \Delta s^2 > 0 \\ \text{null or lightlike} & \text{if } \Delta s^2 = 0 \\ \text{spacelike} & \text{if } \Delta s^2 < 0 \end{array} \tag{12}$$

1.9 Proper time, proper distance

The scalar spacetime squared interval Δs^2 has a physical meaning.

If an interval $\{\Delta t, \Delta \boldsymbol{r}\}$ is timelike, $\Delta t > \Delta r$, then the spacetime interval equals the **proper time** $\Delta\tau$ along it

$$\Delta\tau = \sqrt{\Delta s^2} = \sqrt{\Delta t^2 - \Delta r^2} \ . \tag{13}$$

This is the time experienced by an observer moving along that interval.

If an interval $\{\Delta t, \Delta \boldsymbol{r}\}$ is spacelike, $\Delta t < \Delta r$, then the square root of minus the spacetime interval squared is the **proper distance** Δl along it

$$\Delta l = \sqrt{-\Delta s^2} = \sqrt{\Delta r^2 - \Delta t^2} \ . \tag{14}$$

This is the distance between two events measured by an observer for whom those events are simultaneous.

1.10 Time dilation

If a timelike interval $\{\Delta t, \Delta r\}$ corresponds to motion at velocity v, then $\Delta r = v\Delta t$. The proper time along the interval is

$$\Delta\tau = \sqrt{\Delta t^2 - \Delta r^2} = \Delta t\sqrt{1 - v^2} = \frac{\Delta t}{\gamma} \ . \tag{15}$$

This is Lorentz time dilation: the proper time interval $\Delta\tau$ experienced by a moving person is a factor γ less than the time interval Δt according to an onlooker.

1.11 Lorentz contraction

Consider a rocket of proper length l, so that in the rocket's own rest frame (primed) the back and front ends of the rocket move through time t' with coordinates

$$\{t', x'\} = \{t', 0\} \text{ and } \{t', l\} \ . \tag{16}$$

From the perspective of an observer who sees the rocket move at velocity v in the x-direction, the worldlines of the back and front ends of the rocket are at

$$\{t, x\} = \{\gamma t', \gamma v t'\} \text{ and } \{\gamma t' + \gamma v l, \gamma v t' + \gamma l\} \ . \tag{17}$$

However, the observer measures the length of the rocket simultaneously in their own frame, not the rocket frame. Solving for $\gamma t' = t$ at the back and $\gamma t' + \gamma v l = t$ at the front gives

$$\{t, x\} = \{t, vt\} \text{ and } \left\{t, vt + \frac{l}{\gamma}\right\} \tag{18}$$

which says that the observer measures the front end of the rocket to be a distance l/γ ahead of the back end. This is Lorentz contraction: an object of proper length l is measured by a moving person to be shorter by a factor γ.

1.12 Energy-momentum 4-vector

Symmetry argument:

Symmetry	Conservation law
Time translation	Energy
Space translation	Momentum

suggests

$$\left.\begin{array}{rcl} \text{energy} & = & \text{time component} \\ \text{momentum} & = & \text{space component} \end{array}\right\} \text{ of 4-vector.} \tag{19}$$

The Principle of Special Relativity requires that the equation of energy-momentum conservation

$$\begin{array}{r} \text{energy} \\ \text{momentum} \end{array} = \text{constant} \tag{20}$$

should take the same form in any inertial frame. The equation should be **Lorentz covariant**, that is, the equation should transform like a Lorentz 4-vector.

1.13 Construction of the energy-momentum 4-vector

Require:

1. it's a 4-vector
2. goes over to the Newtonian limit as $v \to 0$.

Newtonian limit:

Momentum $\boldsymbol{p}$ is mass m times velocity $\boldsymbol{v}$

$$\boldsymbol{p} = m\boldsymbol{v} = m\frac{d\boldsymbol{r}}{dt} \ . \tag{21}$$

4D version:

Need to do 2 things to Newtonian momentum:

- replace $\boldsymbol{r}$ by a 4-vector $x^m = \{t, \boldsymbol{r}\}$
- replace dt by a scalar – the only obvious choice is the proper time interval τ.

Result:

$$\begin{aligned} p^k &= m\frac{dx^k}{d\tau} \\ &= m\left\{\frac{dt}{d\tau}, \frac{d\boldsymbol{r}}{d\tau}\right\} \\ &= m\{\gamma, \gamma\boldsymbol{v}\} \end{aligned} \tag{22}$$

which are SR versions of energy E and momentum $\boldsymbol{p}$

$$p^k = \{E, \boldsymbol{p}\} = \{m\gamma, m\gamma\boldsymbol{v}\} \ . \tag{23}$$

1.14 Special relativistic energy

$$E = m\gamma \quad \text{(units } c = 1\text{)} \tag{24}$$

or, restoring standard units

$$E = mc^2\gamma \ . \tag{25}$$

Taylor expand γ for small velocity v:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + ... \tag{26}$$

so

$$\begin{aligned} E &= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + ...\right) \\ &= mc^2 + \frac{1}{2}mv^2 + ... \ . \end{aligned} \tag{27}$$

The first term, mc^2, is the rest-mass energy. The second term, $\frac{1}{2}mv^2$, is the non-relativistic kinetic energy. Higher-order terms give relativistic corrections to the kinetic energy.

1.15 Rest mass is a scalar

The scalar quantity constructed from the energy-momentum 4-vector $p^k = \{E, \boldsymbol{p}\}$ is

$$\begin{aligned} p_k p^k &= E^2 - p^2 \\ &= m^2(\gamma^2 - \gamma^2 v^2) \\ &= m^2 \end{aligned} \tag{28}$$

the square of the rest mass.

1.16 Photon energy-momentum

Photons have zero rest mass

$$m = 0 \ . \tag{29}$$

Thus

$$p_k p^k = E^2 - p^2 = m^2 = 0 \tag{30}$$

whence

$$p \equiv |\boldsymbol{p}| = E \ . \tag{31}$$

Hence

$$\begin{aligned} p^k &= \{E, \boldsymbol{p}\} \\ &= E\{1, \boldsymbol{n}\} \\ &= h\nu\{1, \boldsymbol{n}\} \end{aligned} \tag{32}$$

where ν is the photon frequency.

The photon velocity is $\boldsymbol{n}$, a unit vector. The photon speed is one (the speed of light).

1.17 Lorentz transformation of photon energy-momentum 4-vector

Follows the usual rules for 4-vectors.

In the case that the Lorentz transformation is a Lorentz boost along the x-axis, the transformation is

$$\begin{pmatrix} p'^t \\ p'^x \\ p'^y \\ p'^z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^t \\ p^x \\ p^y \\ p^z \end{pmatrix} = \begin{pmatrix} \gamma(p'^t - v p'^x) \\ \gamma(p'^x - v p'^t) \\ p'^y \\ p'^z \end{pmatrix} . \tag{33}$$

Equivalently

$$h\nu' \begin{pmatrix} 1 \\ n'^x \\ n'^y \\ n'^z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} h\nu \begin{pmatrix} 1 \\ n^x \\ n^y \\ n^z \end{pmatrix} = h\nu \begin{pmatrix} \gamma(1 - n^x v) \\ \gamma(n^x - v) \\ n^y \\ n^z \end{pmatrix} .$$

These mathematical relations imply the Rules of 4-Dimensional Perspective.

1.18 Redshift

Astronomers define the **redshift** z of a photon by

$$z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} . \tag{34}$$

In relativity, it is often more convenient to use the **redshift factor** $1 + z$

$$1 + z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} . \tag{35}$$

1.19 Special relativistic Doppler shift

If the emitter frame (primed) is moving with velocity v in the x-direction relative to the observer frame (unprimed) then

$$h\nu_{\text{emit}} = h\nu_{\text{obs}}\gamma(1 - n^x v) \tag{36}$$

so

$$\begin{aligned} 1 + z &= \frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} \\ &= \gamma(1 - n^x v) \\ &= \gamma(1 - \boldsymbol{n}.\boldsymbol{v}) . \end{aligned} \tag{37}$$

This is the general formula for the special relativistic Doppler shift.