A Guide to Error Propagation

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General Formula

There is a general formula that can be used to propagate errors in any equation. For any quantity f that is a function of variables $u_1, u_2, \ldots, u_n$, you can find the error in f, σ_f, from $\sigma_{u_2}, \ldots, \sigma_{u_n}$ using the following formula:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial u_1}\right)^2 \sigma_{u_1}^2 + \left(\frac{\partial f}{\partial u_2}\right)^2 \sigma_{u_2}^2 + \ldots + \left(\frac{\partial f}{\partial u_n}\right)^2 \sigma_{u_n}^2 \tag{1}$$

Many students don't like to use Equation 1 because they are uncomfortable with partial derivatives. Fortunately, in almost all cases you can still propagate errors just by knowing a couple of simple formulae and being willing to do some algebra.

The Simple Formulae

Addition/Subtraction: If $f = a \pm b$, then σ_f is given by:

$$\sigma_f^2 = \sigma_a^2 + \sigma_b^2 \tag{2}$$

Multiplication/Division: If $f = ab$ or $f = a/b$, then σ_f is given by:

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 \tag{3}$$

Exponentiation: If $f = a^c$, where c is a constant, then σ_f is given by:

$$\frac{\sigma_f}{f} = c\left(\frac{\sigma_a}{a}\right) \tag{4}$$

Example

Let's look at an example of how to use these simple formulae to find an error in a reasonably complex quantity, the distance between two points (d):

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{5}$$

While you can plug Equation 5 directly into Equation 1 to find σ_d if you're willing to evaluate the partial derivatives, you can't plug it into any of our simple formulae (Eqns 2-4) to do the same thing. You can, however, break Equation 5 down into chunks that you can deal with using Equations 2-4 and then put the chunks back together to find σ_d.

In this example, (x_1, y_1) and (x_2, y_2) are positions with known errors $(\sigma_{x_1}, \sigma_{y_1})$ and $(\sigma_{x_2}, \sigma_{y_2})$. In fact, since we can calculate d directly from the positions themselves, the *only* unknown quantity if σ_d. The key to finding σ_d using Eqns 2-4 is to break Equation 5 up into bits that you can plug into the simple formulae, working from the inside out. Start by making the following substitutions:

$$p = x_1 - x_2 \qquad\qquad q = y_1 - y_2 \tag{6}$$

The errors in p and q can be found using Equation 2:

$$\sigma_p^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 \qquad\qquad \sigma_q^2 = \sigma_{y_1}^2 + \sigma_{y_2}^2 \tag{7}$$

With this substitution, Equation 5 becomes:

$$d = \sqrt{p^2 + q^2} \tag{8}$$

We can further simplify Equation 8 with these substitutions:

$$u = p^2 \qquad\qquad v = q^2 \tag{9}$$

The errors in u and v can be found using Equation 4:

$$\frac{\sigma_u}{u} = 2\left(\frac{\sigma_p}{p}\right) \qquad\qquad \frac{\sigma_v}{v} = 2\left(\frac{\sigma_q}{q}\right) \tag{10}$$

Equation 8 now reduces to:

$$d = \sqrt{u + v} \tag{11}$$

Again we can simplify, this time using the substitution:

$$w = u + v \tag{12}$$

The error in w is easily found from Equation 2:

$$\sigma_w^2 = \sigma_u^2 + \sigma_v^2 \tag{13}$$

Equation 5 can now be reduced to its simplest form:

$$d = \sqrt{w} \tag{14}$$

We've now simplified things enough to find σ_d directly from Equation 4:

$$\frac{\sigma_d}{d} = \frac{1}{2}\left(\frac{\sigma_w}{w}\right) \tag{15}$$

Now all that's left is to use Equations 7, 10, 13 & 15 to get σ_d in terms of our original variables:

$$\begin{aligned}
\sigma_d &= \tfrac{1}{2}\tfrac{d}{w}\sigma_w && (16)\\
&= \tfrac{1}{2d}\sigma_w && (17)\\
&= \tfrac{1}{2d}\sqrt{\sigma_u^2 + \sigma_v^2} && (18)\\
&= \tfrac{1}{2d}\sqrt{4(\tfrac{u}{p})^2\sigma_p^2 + 4(\tfrac{v}{q})^2\sigma_q^2} && (19)\\
&= d^{-1}\sqrt{p^2\sigma_p^2 + q^2\sigma_q^2} && (20)\\
&= d^{-1}\sqrt{(x_1 - x_2)^2(\sigma_{x_1}^2 + \sigma_{x_2}^2) + (y_1 - y_2)^2(\sigma_{y_1}^2 + \sigma_{y_2}^2)} && (21)
\end{aligned}$$

I warned you that it was a lot of algebra, but if you understand the example above, then you are in good shape. Good luck on the lab and, as always, if you have any questions feel free to ask!