

8 What's All the Fuss About Metacognition?

Alan H. Schoenfeld
Education and Mathematics
The University of California, Berkeley

This chapter was written after the second of our two conferences, in response to a challenge from (among others) Joe Crosswhite, Henry Pollak, Anna Henderson, and Steve Maurer. You'll find their comments in various places throughout this book. Their challenge can be summarized as follows:

Metacognition is a buzzword for you researchers. Over the past few days [of the conference] we've heard about metacognitive this, metacognitive that, metacognitive the other. The word has been used in almost every talk, and almost every panel discussion, since we got started. But the plain fact is that it's jargon and doesn't communicate anything to us nonresearchers. If metacognition is so important, you have a responsibility to explain to us (a) what it is, (b) why it's important, and (c) what to do about it—all in clear language that we can understand.

What follows is an attempt to do just that. The next section discusses (a) and (b) together, defining metacognition and explaining why it is worth worrying about. The section that follows deals with (c), describing some of the classroom techniques I use to help students develop the "right" metacognitive skills.

WHAT METACOGNITION IS AND WHY IT'S IMPORTANT

Translating the term "metacognition" into everyday language, one gets something like "reflections on cognition" or "thinking about your own thinking." Although those definitions are in the ballpark, they're not

Handwritten note: + mind on society
Feb 6, 1980

Handwritten note: in Sci + Math Ed.
Cog. Ed. BY SCHOENFELD

precise enough to be useful. More precisely, research on metacognition has focused on three related but distinct categories of intellectual behavior:

1. Your knowledge about your own thought processes. How accurate are you in describing your own thinking?
2. Control, or self-regulation. How well do you keep track of what you're doing when (for example) you're solving problems, and how well (if at all) do you use the input from those observations to guide your problem solving actions?
3. Beliefs and intuitions. What ideas about mathematics do you bring to your work in mathematics, and how does that shape the way that you do mathematics?

There is a large body of fascinating research on the first item. Because most of the work with direct implications for mathematics educators has been focused on the second and third categories, however, we are going to skim briefly over the first. To sum it up in a few words, the research indicates that children are not very good at describing their own mental abilities, but that they get better (though nowhere near perfect) as they get older. The largest body of research is on "metamemory," or people's ability to describe how good they are at remembering things. Young children have very little idea of how well they can memorize. Although they may say (and believe) that they can easily memorize a hundred unrelated words, in fact they will have trouble memorizing more than four or five. As children get older, their estimates of their memory skills become more and more accurate.

Why is work in this area so important? For one thing, we're interested in helping students develop good study skills. Those skills depend, in part, on students' ability to make realistic assessments of what they can learn. Hence, it is important to know how likely it is that students will reflect on their thinking and how accurate those reflections will be. Similarly, good problem solving calls for using efficiently what you know; if you don't have a good sense of what you know, you may find it difficult to be an efficient problem solver. (I shall have no more to say about this aspect of metacognition in this chapter. For those who wish to pursue it, two good entry points into the literature are articles by Brown, 1978, and Brown, Bransford, Ferrara, and Campione, 1983.)

We turn to the second aspect of metacognition, control, or self-regulation. Another way to look at this aspect of metacognition is to think of it as a management issue: How well do you manage your time and effort as you are working on complex tasks? Aspects of management include (a) making sure that you understand what a problem is all about

before you hastily attempt a solution; (b) planning; (c) monitoring, or keeping track of how well things are going during a solution; and (d) allocating resources, or deciding what to do, and for how long, as you work on the problem. Let me begin with two brief introductory examples and then broaden the discussion. The first example comes from calculus, the second from algebra.

A number of years ago I prepared an examination on techniques of integration for a calculus class. The exam began with the following problem:

$$\int \frac{x}{x^2 - 9} dx$$

I had chosen this problem quite carefully, expecting it to be a "confidence builder" for the students. If you note the relationship between numerator and denominator (the numerator is $\frac{1}{2}$ the derivative of the denominator), you can rewrite this problem as

$$\frac{1}{2} \int \frac{2x}{x^2 - 9} dx,$$

and solve it in one step using the substitution $u = (x^2 - 9)$. Substitution being the most basic technique of integration, the students certainly knew how to use it. I expected them to solve the problem in 2 minutes or less, feel good about it, and move on with confidence to the rest of the exam.

The exam was given to a large lecture class. About half the students did what I had hoped. But nearly one fourth of the students solved the problem using a technique called "partial fractions," in which they had to do some fairly complicated algebra to re-express the fraction $x/(x^2 - 9)$ in the form $A/(x - 3) + B/(x + 3)$. This technique works, but it takes most students at least 5 and perhaps 10 minutes to carry it out. But the time these students finished the first problem, they were behind schedule and scrambling to finish the test. Even worse, more than one ninth of the students solved the problem by using the trigonometric substitution $x = 3\sin\theta$. This substitution also works—but it takes most students 10–15 minutes to use it. The students who did so wound up very far behind and did quite poorly on the test.

The point of these examples is not that the students failed to understand the mathematics. If anything, the students who used partial fractions and trig substitutions demonstrated mastery of more difficult subject matter than did the ones who used the simple substitution. The issue here is not what they knew—it's what they decided to use. The students who used partial fractions and trig substitutions violated a cardinal rule of problem solving: *Never use any difficult techniques before checking to see whether simple techniques will do the job.* Had the students asked

themselves if there might be an easy way to do the problem, they might have avoided a great deal of unnecessary work.

I shall not give all the gory details of my algebra example; a brief description is enough for all mathematics teachers to recognize the behavior it discusses. In a typical videotaping session, some students were asked to solve the algebraic equation

$$x^2y + 2xy + x - y = 2$$

for y . They began by moving the lone y over to the right-hand side of the equation, which seems reasonable. They brought together the y terms on the left-hand side of the equation, factored out the common y , and then factored the common x term from that expression as well. Then they moved some terms over to the right-hand side . . . and so on. Each manipulation was correct. But a close look at what they did reveals that the equation they were working on after six algebraic steps was more complicated than the one they had started with! They simply proceeded one step at a time. Each time they got a new equation they seemed to start over: "Now what can we do with *this* equation?" As a result, they dug themselves into a deeper and deeper hole, without standing back to see if what they were doing made sense. This kind of behavior occurs all too frequently. (For a more extended treatment of students' algebraic foibles, see Ron Wenger's discussion in chapter 9.)

Both of these examples illustrate the main point about self-regulation: It's not only what you know, but how you use it (if at all) that matters. The discussion of Figs. 8.1 and 8.2 provide dramatic illustrations of this point. These discussions provide summary outlines of two representative problem sessions. Readers who would like all the details can find them in Chapters 4 and 9 of my (1985) *Mathematical Problem Solving*.

In my problem-solving research I frequently make videotapes of students solving problems, and then analyze their solutions in detail. Figure 8.1 presents the graph of an attempt by two students to solve a reasonably difficult problem: finding the largest triangle that can be inscribed in a circle. As it happens, this is a standard "max-min" problem, just like one that one of the students had solved correctly on a calculus final exam the week before. (He got an A in the course.) Both students knew more than enough mathematics to solve the problem without difficulty. But this problem was given out of context, and that made a big difference.

In brief, this is what happened. The students read the problem, made a conjecture (the correct one, that the largest triangle is equilateral), and then set out to calculate the area of the equilateral triangle. They made some mistakes and got bogged down in the calculations. They were still enmeshed in those calculations 20 minutes later, when the video equipment recording them clicked off. At that point, I told them the area of the

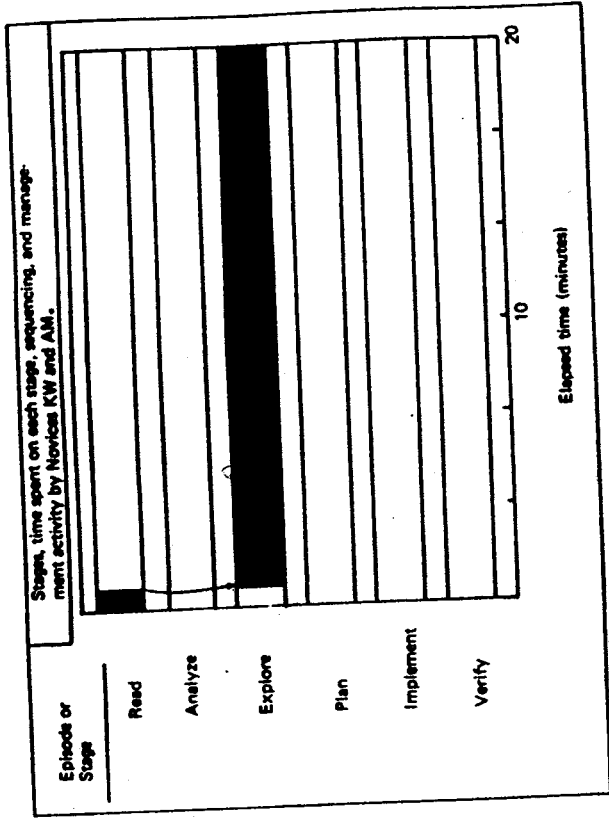


FIG. 8.1.

triangle and asked how knowing it would help them solve the original problem. They couldn't tell me.

The students had spent twenty minutes on a wild goose chase. They had ample opportunity to stop during that time and ask themselves "Is this getting us anywhere? Should we try something else?" but they didn't. And as long as they didn't, they were guaranteed not to solve the problem.

This is an all too typical example of the disastrous consequences of an absence of self-regulation. In an extensive series of videotaping sessions where students were recorded solving nonstandard problems outside the classroom, I found that more than half the problem sessions were of the type illustrated in Fig. 8.1, where students read the problem, decided to do something, and then kept at it to the exclusion of all other possibilities. The students were remarkably consistent—and persistent—in pursuing wild mathematical geese. Because I was familiar with their backgrounds, I had evidence that they knew the mathematics that would provide solutions to the problems they worked. Because of their wild goose chases, however, they never got around to using it. The result was an extremely high failure rate.

Figure 8.2, which shows the graph of an attempt by a mathematician to solve a difficult geometry problem, provides an interesting contrast. The

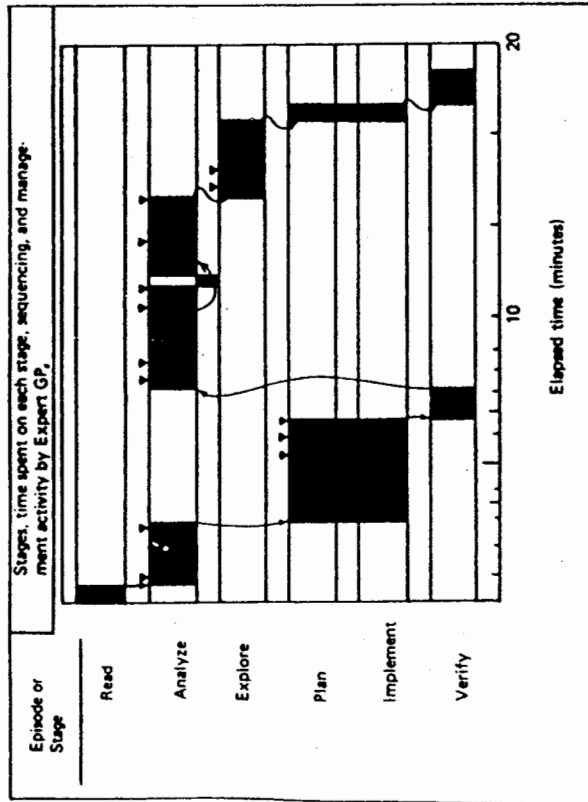


FIG. 8.2.

mathematician had not worked in geometry for a number of years, and he was pretty rusty when he started working the problem. He was far from a geometry expert, but he was an expert at problem solving. Interesting about his solution is that as he worked his way through the problem, he considered quite a few different approaches to it, many of which were just plain wrong. In fact, he generated enough potential wild geese to keep dozens of problem solvers busy. Yet he never went off the deep end as the students had, because he was as ruthless about testing and rejecting ideas as he was ingenious in generating them.

The problem that the mathematician worked had two parts, which explains the bipartite nature of the graph in Fig. 8.2. One difference (besides the fact that he solved the problem) between this graph and the graph of the students' work is immediately apparent: The mathematician spent the vast majority of his time *thinking* rather than *doing*. More precisely, he spent a great deal of time analyzing the problem. When he had made sense of it, working out the details of the solution didn't take long at all. The second feature distinguishing between the graphs is that there are a large number of small inverted triangles in Fig. 8.2, and none at all in Fig. 8.1. Each of those triangles indicates a time when the problem solver effectively pulled up short, asked "How am I doing?" and decided what to do as a result. The two marks at the beginning and end of

the first "analyze" phase, for example, correspond respectively to comments about the need to make sense of the problem and the fact that (after analysis) he understood it fairly well and was ready to try out a solution. The comments near the end of the first plan/implement phase had to do with his being near a solution and the need to check it. He did solve the problem, and he checked his solution. When he began working on the second part of the problem he commented that it looked tricky and that he'd better be careful. That turned out to be true. He started off on a wild goose chase, but—and this is absolutely critical—he curtailed it quickly ("But I don't like that. It doesn't seem the way to go."), changed direction, and went on to find a solution.

In sum, the difference between the mathematician's success and the students' failure cannot be attributed to a difference in knowledge of subject matter. Indeed, the students started off with a clear advantage over the mathematician. They knew all of the procedures required to solve the problem they were given, whereas he did not remember them and had to figure them out for himself. What made the difference was how the problem solvers made use of what they did know. The students decided to try something and went off on a wild goose chase, never to return. The mathematician tried many approaches, but only briefly if they didn't seem to work. With the efficient use of self-monitoring and self-regulation, he solved a problem that many students—who knew a lot more geometry than he did—failed to solve.

We now turn to the third aspect of metacognition, beliefs and intuitions. These ideas were introduced in our discussion of constructivism in chapter 1. To recapitulate, people are *interpreters* of the world around them. They don't necessarily see "what's out there"—some version of "objective reality"—but instead perceive what they experience in the light of interpretive frameworks they have developed. As we said in chapter 1, this has significant implications for our teaching. When we try to teach some new subject matter, we can't assume that our students are empty containers waiting to be filled with knowledge. The students may have preconceptions and misconceptions about much of the subject matter they study, and we would do well to take that into account. Moreover, as the literature on bugs indicates, students may consistently misinterpret the procedures they learn in our classrooms.

The kinds of beliefs discussed here are more subtle, and more distressing. I argue that, largely as a result of their instruction, many students develop some beliefs about "what mathematics is all about" that are just plain wrong—and that those beliefs have a very strong negative effect on the students' mathematical behavior. Two examples illustrate the point.

The first example comes from the third National Assessment of Educational Progress (Carpenter, Lindquist, Matthews, & Silver, 1983). One of

the problems on the NAEP secondary mathematics exam, which was administered to a stratified sample of 45,000 students nationwide, was the following: An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site. How many buses are needed?

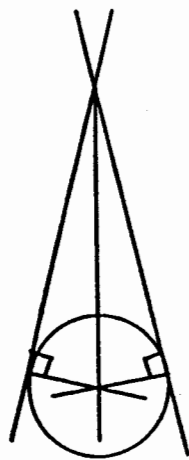
Seventy percent of the students who took the exam set up the correct long division and performed it correctly. However, the following are the answers those students gave to the question of *how many buses are needed*: 29% said the number of buses needed is "31 remainder 12"; 18% said the number of buses needed is "31"; 23% said the number of buses needed is "32," which is correct; (30% did not do the computation correctly).

It's frightening enough that fewer than one fourth of the students got the right answer. More frightening is that almost one out of three students said that the number of buses needed is "31 remainder 12." Those students picked some numbers and an operation from the problem, did the computation, and wrote down the answer—without checking (or at least, without checking carefully) to see if the result made sense! In essence, they treated the problem as calling for a formal computation. Despite the "cover story" about the buses, the computation had little or nothing to do with the real world.

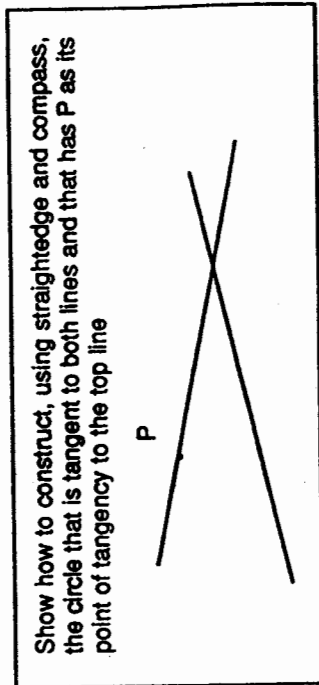
The second example comes from my research in geometry. I have given students a series of geometry problems, in which they were asked (a) to prove that certain geometrical figures have certain properties, and (b) to construct those figures. In a proof problem, for example, the students show that the center of a given circle lies at the intersection of two particular line segments. The students are then given the same diagram without the circle and are asked how to construct the circle. Of course, this second problem is not a problem at all: the proof states what properties the circle must have and therefore how to construct it. Yet about 30% of the students who solve the proof problem correctly then make a conjecture that flatly violates what they have just proven (Fig. 8.3). In other words, they ignore the results of the proof when working the construction problem. Why? These students see little or no connection between the two problems. From their point of view, proof problems confirm what they already know or what they have been told is true. Construction problems ask you to find something. Thus, when they work construction problems and are in "discovery mode," the results of proof—or "confirmation mode"—are simply irrelevant.

Much of my research in recent years has been devoted to exploring students' beliefs about mathematics, and I have spent a great deal of time making classroom observations to find the origins of those beliefs. Details may be found in my (1985) *Mathematical Problem Solving* and in my (in press) article, "When good teaching leads to bad results." Some of the unpleasant findings are as follows.

Despite having proved — using the formal tools of Euclidean Geometry — that the center of the circle tangent to two given lines lies at the intersection of the two perpendiculars and the angle bisector indicated in the following diagram,



Thirty percent of the students asked to solve this problem:



conjectured that the center of the circle was the midpoint of the line segment that joined P and its "opposite," P'.

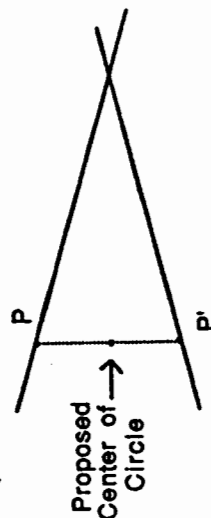


FIG. 8.3.

Many students come to believe that school mathematics consists of mastering formal procedures that are completely divorced from real life, from discovery, and from problem solving. This is the kind of belief behind the behavior in the two examples just discussed. After all, if mathematics problems were meaningful to students, could students possibly report that the number of buses they need is 31-remainder-12? (A

colleague has pointed out that a group of students in the schoolyard, deciding how many cars they needed to go someplace, would never make the same mistake.) Similarly, students disregard proof because it's meaningless to them. Some other student beliefs—stated harshly to be provocative—are as follows: All problems can be solved in 10 minutes or less; the form of a mathematical argument is more important than its correctness; only geniuses are capable of discovering mathematics.

These beliefs, like the one discussed earlier, have unfortunate behavioral consequences. Students who believe that all problems can be solved in ten minutes or less will simply stop working on a problem after a few minutes, even if they would have been able to solve it with more effort. Students concerned with form (e.g., writing two-column proofs in geometry with correct abbreviations such as "CPCIT") will spend more time worrying about the form of their answer than they will trying to understand the result they are writing down. And students who believe that mathematical understanding is simply beyond ordinary mortals like themselves become passive consumers of mathematics, accepting and memorizing what is handed to them without attempting to make sense of it on their own.

These examples show that beliefs and intuitions, like self-awareness and self-regulation, are important determinants of students' mathematical behavior. "Knowing" a lot of mathematics may not do students much good if their beliefs keep them from using it. Moreover, students who lack good self-regulation skills still may go off on wild goose chases and never have the opportunity to exploit what they have learned. In sum, meta-cognition deserves our attention. Now, what do we do about it?

A "KITCHEN SINK" APPROACH TO DEVELOPING METACOGNITIVE SKILLS

This section describes four classroom techniques that focus on meta-cognition, and the rationales for them. The techniques were developed in my problem-solving courses, and (since I have a lot of flexibility there) it's easiest to use them in that setting. However, all of them can be adapted for use in virtually any mathematics instruction. We start with the technique that is least "interventionist" and proceed to techniques that call for increasingly deeper interactions between the teacher and students.

First Technique: Using Videotapes

Students asked what they do when they work on problems typically will say that they do "what comes to mind"—as though their minds are independent, autonomous entities over which they have little or no

control. Of course this last statement is an exaggeration, but only a small one. The fact is that most students are largely unaware of their thinking processes. Virtually none of the students who enter my problem-solving classes are aware that they can practice their thinking skills and get better at them. Yet self-awareness is a crucial aspect of metacognition, for awareness of one's intellectual behavior is a prerequisite for working to change it. For that reason I make it a point to bring the subject out in the open.

Early in the term I describe the kinds of things we'll be doing during the term and why we'll be doing them. This introduction to the course includes a discussion of problem-solving strategies and of issues such as self-regulation and belief. My job, in part, is to convince the students of the importance of these aspects of mathematical thinking.

It's easy to make a good case for the problem-solving strategies to be studied in the course. The first day of class I give the class some carefully chosen problems to work on. I know from experience that the students could work on these problems for hours without success—but that the problems can be "unlocked" by some simple hints such as "Try the values of $n = 1, 2, 3, 4$, and look for a pattern" or "Try drawing a picture." I let the class work on the problems for quite a while, and when they have run out of ideas I offer the hints. With the hints (which are general suggestions, and not problem specific), the students solve the problems in 3 minutes—and they are convinced that I have something to teach them. This is enough to induce them to try the strategies I introduce throughout the term. (Of course, I also tell them that these are specially chosen problems; very few problems can be solved so easily by the application of just one strategy.)

Self-awareness is a more difficult idea to get across. After all, these students are mathematically talented; they're in college precisely because their thinking habits have enabled them to be successful. Yet here I am telling them that they're not very efficient thinkers at all and that they could do a lot better. They won't accept this statement at face value, and they shouldn't: The burden of proof is on me. One way I make the case convincingly is to show the students videotapes of other students working on problems, for example, the videotape that produced the graph in Fig. 8.1. The students' reaction is interesting. On one hand, they become quite upset looking at the tapes: "What they're doing is stupid! They're wasting all that time, and it won't do them any good!" On the other hand, they are embarrassed because they empathize with the students: "That could be me!" That's precisely the point. It's a lot easier to analyze behavior when it's someone else's, and then to see that the analysis applies to yourself. As a result of watching these tapes and discussing them, the students are made aware of metacognitive issues. They are then more receptive to some of the more interventionist techniques I use later in the term.

Second Technique: Teacher as Role Model for Metacognitive Behavior

Like all teachers, I spend a fair amount of time presenting problem solutions at the blackboard—although I try to keep my “straight” presentations to a minimum, to make time for the alternative class formats to be described below. When I do present solutions, however, I try to do more than just demonstrate the “right” way to get an answer.

When we write the solution to a problem on the board in class, we usually present the results of our thinking in a neat and clean presentation of the answer that we’ve worked out. In fact, presenting things neatly is part of our professionalism. We may work for quite a while to solve a difficult problem, but the idea is to “get it right” and then present a polished product. Unfortunately, this professionalism has an unintended byproduct. In presenting a polished solution, we often obscure the processes that yielded it, thus giving the impression that things should be easy for people who student the subject matter. In consequence, the give-and-take of real problem solving—the false starts, the recoveries from them, the interesting insights, and the ways we capitalize on them, and so on—are all hidden from students. Yet these are the processes that must be brought out into the open.

One way to bring them out in the open is to model them, presenting “problem resolutions” rather than problem solutions. At times I work a problem as though I were working it from scratch, going blow-by-blow through the solution process. That may mean looking at a few examples to make sure that I understand the problem. Then I may make a few tentative explorations, looking for promising things to do. If I generate a few reasonable approaches, I decide among them and pursue one for a while. After a few minutes of working on it, I reconsider (“Am I making reasonable progress? Does this seem like the right thing to do?”) and then act accordingly. I may find a solution along the lines I first pursued, or I may have to back off and try something else before succeeding. This continues until I solve the problem, at which point I do a “post mortem” and review the whole solution.¹

I am the first to admit that this kind of modeling approach is artificial and must be used sparingly, for it wears thin pretty rapidly. Like the videotape sessions described earlier, these simulated problem resolutions are not intended for extended use. Rather, their primary function is to focus students’ attention on metacognitive behaviors. Both methods bring

¹For a detailed description of one such problem resolution, see pp. 8–13 of my (1983) *Problem Solving in the Mathematics Curriculum*. For a far more extensive treatment, see John Mason, Leone Burton, and Kaye Stacey’s (1982) *Thinking Mathematically*. That book presents an extended discussion of the give-and-take of problem solving, including “multi-ling” and what to do when you get “stuck.”

those behaviors to center stage, highlighting their importance and preparing the students for more direct work on their metacognitive skills. Such work is the focus of techniques three and four.

Third Technique: Whole-Class Discussions of Problems with Teacher Serving as “Control”

The vast majority of time in my problem-solving classes, and as much time as I can arrange in all of my other classes, is spent with students actively working problems and discussing their solutions. I use two formats for these problem-solving sessions, small-group (discussed in the next section) and whole-class.

When the class works as a whole on problems, I take the role of scribe and orchestrator of the students’ suggestions. I do not try to guide the students to the correct solutions, based on my knowledge of the mathematics. This is the standard technique used in “Socratic dialogues”—a technique of some value, but not appropriate for the goals I have in mind. Rather, my task is to help the students make the most of what they themselves generate and to help them reflect on how they do it.

Problem sessions begin when I hand out a list of new problems or ask if any of the homework problems merit class discussion. I write the problem to be discussed on the board and ask if there are any suggestions for solving it. Often one student has an “inspiration” and within a few seconds of reading the problem suggests “Let’s try X.” At times X is reasonable, and at times not. My task is not to say yes or no, or even to evaluate the suggestion. Rather, it is to raise the issue for discussion. If (as is frequently the case at the beginning of the term) the suggestion has come with breathtaking quickness, I often respond as follows: “Before we try this suggestion, is everyone sure he understands the problem?” Typically a number of students respond no, and I ask how we can make sense of the problem. We then do whatever seems reasonable: exploring the problem conditions, drawing a diagram, working some examples, and so on. When we get to the point where we understand the problem, I return to the original suggestion. “All right, what about X? Is that what we want to do?”

If the original suggestion was inappropriate, our discussions often reveal it to be so: When we’ve made sense of the problem, the suggestion simply doesn’t make sense. As a result of our discussions, the student who had suggested X often realizes that it is not useful and retracts the suggestion on his own. When this happens, I step out of my role as moderator to make the point to the whole class: If you make sure you understand the problem before you jump into a solution, you are less likely to go off on a wild goose chase.

NOT

Takes
2.4.0
PBC

Regardless of the appropriateness of the first suggestion (and whether or not it has been retracted), I ask for other suggestions. Typically there are three or four approaches we might try. As moderator, I ask for a discussion of which one we should attempt, and why. The class makes its decision, and we begin carrying it out. The class may have chosen a direction that leads to a solution, or they may have decided to go down what I know to be a blind alley; either one is fine. I make no attempt to lead the class in the right direction, for what matters is that they make their decision reasonably. Once they have, we work on it.

Whether or not things are going well from my perspective, we stop for a midstream review about 5 minutes into the solution. "We've been doing this for 5 minutes or so. Do things seem to be going pretty well? If so, we should continue. But if not, we might want to reconsider." (Note that it is important to ask these questions even if things are going well; if you use them only when things have gone wrong, students learn to interpret them as a prompt to change directions. The idea is that one should *always* be concerned about the progress of a solution and always ready to reconsider if it seems like things have gotten bogged down.) If the class decides to pursue the direction we have been working in, we continue accordingly. If it decides to abandon it, I ask if anything can be salvaged from what we've done so far: "Are there ideas we might want to return to, or things we might want to try if our new approach doesn't pan out?" We continue along these lines until we reach a solution, at which point I step out of the moderator's role to do a debriefing. There I summarize what the class has done and comment on the efficiency of the solution, pointing out where things have gone awry or where the class might have taken advantage of something they failed to exploit. I also discuss other approaches for working the problem that were advanced during the class's attempt to solve it. We often return to these, producing alternative solutions to the problem.

Whole-class problem sessions like the one just described are an obvious vehicle for dealing with issues of self-regulation. In a sense, self-regulation is the focus of these whole-class sessions. My presence as moderator forces the class to focus on control decisions. However, I am not the one making those decisions: The class itself chooses what to pursue.² And although this is a class activity, it works very well for individual students. First, the discussions are out in the open and are later analyzed for their efficiency. This provides the opportunity to reflect on self-regulation and how it works. Second, there is a shared burden; all

²See pp. 42-50 of my (1983) *Problem Solving in the Mathematics Curriculum* for a blow-by-blow description of a whole-class problem session, in which the class took about forty minutes to solve a difficult construction problem.

students work as a team rather than doing all the work themselves. Because I'm carrying out their decisions at the board, the students can concentrate on decision-making. And with the group mind working toward a solution, no individual student is responsible for generating all of the ideas or for keeping track of all the options. Yet the students do participate actively. In essence they are apprentice problem solvers, working in a real problem-solving context but bearing responsibility for only part of the task they are working on. Like apprentices to a craft, they take over more and more of the task as they gain experience and expertise. (For an excellent discussion of the apprenticeship model of teaching, see Collins and Brown, in press.)

Equally important, the whole-class problem session is an excellent context for discussing students' beliefs about mathematics. Some counterproductive student beliefs were mentioned in the first part of this chapter. In the whole-class problem sessions I can pose problems that evoke the beliefs, and then discuss them with the class. One typical student belief, for example, is that if you really understand the subject matter, then any assigned problem can be solved in relatively short order. In my courses I assign problems that may take the class a few days or even a few weeks to solve—and I let them know (a) that it will take us that long to arrive at a solution and (b) that that situation is perfectly natural, and much more the norm than arriving at solutions in just a few minutes. (I also give 2-week take-home examinations, with the warning that any student who waits until the second week to begin working on the problems is likely to find himself in dire difficulty.)

Problems from geometry provide a nice context for dealing with the beliefs about formal systems just mentioned—that proof has nothing to do with discovery or invention and that the form of a mathematical argument is more important than its content. Chapter 1 of Polya's (1981) *Mathematical Discovery* is an exceptionally rich source of geometric construction problems, and many of the problems we work are taken from there. In a typical problem, students are asked to construct a problem if they are given some of its "parts"—for example, to construct the triangle that has sides a , b , and c , if you are given line segments whose lengths respectively are the lengths of a , b , and c . This problem is trivial, of course, but its solution leads to interesting discussions. I often assign this as the class's first construction problem, and students usually produce a solution within a minute or two. But when I ask them if they're sure they've got the original triangle (what if I laid down c first, then marked off a and b ?), it takes a bit of thought. Eventually they argue that all the triangles gotten by constructions, and any other triangle with sides a , b , and c , must be identical—they're congruent. Next I ask the class to tell me how to bisect an angle, and again the class produces the right construction (from

memory). But when I ask them why the construction does indeed bisect the angle, the result is often consternation. One year I let the class address this issue for more than half an hour, without any intervention from me whatsoever. Eventually they convinced themselves that the standard construction creates two congruent triangles and that corresponding angles in those triangles were the two "halves" of the angle that they wished to bisect. At that point a student raised his hand and asked, "Are you trying to tell us that congruence is good for something?"

Indeed I was, and the student's question provided the jumping off point for my discussion. But more important than my discussion was the experience that the students had doing the mathematics and reflecting on it. In working the construction problems, the students learned that they could often *derive needed information, even when working on what were apparently "discovery" problems*. For example, if you want to construct the inscribed circle in a given triangle, you can analyze the properties of the circle. When you discover that the center of the circle lies on the bisectors of the three angles of the triangle, you have identified the information that allows you to construct the circle. This example, too, is fairly straightforward. However, others are quite challenging. Problem 1.19 from Pólya's (1981) *Mathematical Discovery* asks you to construct a triangle, given: the length of one side of the triangle, a ; the radius of the inscribed circle, r ; and the measure of the angle A opposite the side a . This problem occupied the class for the better part of a day. In the process of solving it, we discovered a new theorem, and that theorem allowed us to finish the construction.³ By using deduction in this way, students came to appreciate its value. By virtue of my boardwork and the classroom discussions, they also came to understand that the form of their argument (e.g., whether or not a proof was written in two-column format) was not important: What mattered was that their argument had to be complete, logical, and coherently argued.

Another perspective on what I do in the group context is suggested by Collins and Brown's (in press) notion of apprenticeship. Students enter my courses with a set of beliefs about mathematics, beliefs abstracted from their previous experiences with mathematics. That previous experience was, for the most part, training for mastery: The students were given training in ready-made, prepackaged mathematical procedures. In the traditional sense, apprenticeship does much more than provide training; it provides an initiation into a culture. (Consider, for example, the contrast between (a) taking a journalism class and (b) spending a summer doing

³For a more extended discussion of the relationship between deduction and discovery, see my (1986) article, "On having and using geometric knowledge."

entry-level work for a newspaper.) When you learn about a discipline through apprenticeship into it, you are less likely to pick up the kinds of incorrect and counterproductive beliefs about it described earlier. Thus, a significant part of what I attempt to do (in my problem-solving courses in particular, but increasingly in all of my mathematics instruction) is to create a microcosm of mathematical culture—an environment in which my students create and discuss mathematics much as mathematicians do. Having experienced mathematics in this way, students are more likely to develop a more accurate view of what mathematics is and how it is done.

Fourth Technique: Problem Solving in Small Groups

About a third of the time, the students in my problem-solving courses break into groups of three or four to work together on problems that have just been handed out. While they are working on the problems, I move from group to group, answering questions and offering advice.

I began this practice some 10 years ago, on the basis of common sense and a metaphor. To begin with, it just didn't seem right to conduct a problem-solving class solely in lecture or recitation mode. Common sense dictated that, at least some percentage of the time, students in a problem-solving course should be actively engaged in solving problems. But at the heart of the matter—and the heart of the metaphor—was how I defined my role as a teacher. In standard courses, teachers tend to think of themselves as purveyors of information: "Here's what's known, and here's how it's used." Of course delivering such information was a component of my problem-solving courses. I demonstrated various heuristic strategies, for example. But a large part of my task was to help the students become good problem solvers, by helping them to make efficient use of what they already knew. In that sense, my task was more like that of a coach, an "intellectual coach"—to be sure, but one who spends a lot of time working with students to help them improve their problem solving ability.

The coaching metaphor cast my instruction in a different light. Sports coaches spend a fair amount of time showing their students "how to do it right," of course. A tennis coach will demonstrate a good serve, a swimming coach the right way to hold one's arm for the butterfly, and so forth. But coaches who stop at that won't keep their jobs for very long. (Imagine a swimming coach who demonstrates a particular stroke and then says, "Your homework is to practice this stroke. There will be a test on Friday before the meet.") Coaches watch students as they practice and as they compete. They make corrections "on line," because this is the context in which the coaching will have an effect. Indeed, coaches

videotape their students as they practice and go over the tapes with the students in slow motion replays, because that kind of microscopic analysis can help reveal what's working and what's going wrong.

I had a similar responsibility to my students. My goal was to provide them with a variety of problem-solving techniques (various heuristics) and then to coach them in the efficient use of those techniques. If I showed the students the techniques and then said "Go home and practice," I would be missing the opportunity to intervene when it mattered most—when the students were in the midst of problem solving. Indeed, lectures on "monitoring your solution" and "not going off on wild goose chases" are almost guaranteed to have no effect unless there is in-class problem solving. Students who hear me talk about self-regulation but then go back to their dormitory room to do their homework will almost certainly fall right back into their old habits. Such old habits die hard (especially if the students have been successful with them!), and a few words of warning from a teacher are unlikely to change them.

In fact, the students' behavior is unlikely to be affected unless, at least at the beginning, the coaching is fairly pointed. In handouts and my introductory discussion at the beginning of the course, I inform the students that I reserve the right to ask them the following questions⁴ at any time during their problem solving:

- What (exactly) are you doing? (Can you describe it precisely?)
- Why are you doing it? (How does it fit into the solution?)
- How does it help you? (What will you do with the outcome when you obtain it?)

About 2 weeks into the course, I start asking the small groups these questions as I move through the room in my role as problem-solving consultant. I do so gently, reminding the students that I had warned them about this aspect of the course. At first the reaction from each small group is an embarrassed silence. (More precisely, the students can usually answer the first of the three questions, but they have no answers for the second and third.) With apologies, I persevere.

Soon the students realize that I'm serious about the questions and that I will continue to ask them even though doing so makes them feel uncomfortable. To defend themselves against these intrusions, they begin

⁴These questions might leave the impression that I expect the problem solver always to be "on task," knowing exactly what he's doing and why. That's not the case. I'm not, and I don't expect my students to be. One perfectly good answer to the first question, for example, is "I'm mucking around looking for inspiration, and I intend to do so for another 5 minutes." As long as the student is aware of what he's doing, the odds are that he won't continue mucking around indefinitely.

to prepare answers to the questions in advance. Over the course of the semester, the students get in the habit of discussing the questions, both at the beginning of problem sessions and at major decision points during problem solutions. When things work well, discussions of the underlying issues—whether what the students are doing is reasonable and whether they are making progress—become a matter of practice, and there is less and less need to invoke the questions themselves.

I need to back off for a moment. The preceding discussion may be laying undue emphasis on those questions and, correspondingly, on the way that the questions and my interference, shape the group dynamics. In fact, each group of students functions on its own about 95% of the time. In a class of 20–30 students, there are usually between six and eight working groups. At best (or worst, depending on your point of view), I only have between one sixth to one eighth of the time to spend with any group. In practice, I spend less time. By the middle of the term, the group dynamics are well established. I hand out a new set of problems, and the class gets to work. Often I return to my desk, or leave the room for a few minutes. If a group requests my help as a consultant, I work with them for a while.⁵ Then I move to another group, watching their work for a while and asking if things are going well. If the answer is yes, I move on. Often I return to my desk, and the groups continue working alone until I get the feeling that they've reached closure. Then we convene as a "committee of the whole" to discuss the progress they have made.

Does all of the attention to metacognitive issues in my courses pay off? One must be careful in answering such a question, because it's not always clear what pays off in what ways. That is, a course may produce certain kinds of results, but it's dangerous to attribute particular results to any particular aspect of the course. (Indeed, if you take the "cultural transmission" point of view explored later, it may be impossible to do so.) But there is clear evidence that the course produced marked shifts in the students' problem-solving behavior, particularly at the metacognitive level.

I reported earlier that more than half (actually closer to 60%) of the problem-solving sessions I have analyzed are of the "read the problem, make a decision to do something, and then pursue it come hell or high water" type illustrated in Fig. 8.1. After taking my problem-solving courses, fewer than 20% of my students' solution attempts are of that

⁵As the term progresses there are fewer requests for my assistance. This happens both because of acculturation and because of the evolution of the group dynamics. First, the students come to realize that they are not engaged in a contest where the goal is to get the right answer and move on. Their goal (supported by the environment) is to make sense of the mathematics. Second, they realize that it's much better, and more fun, for them to make sense of the problems by themselves.

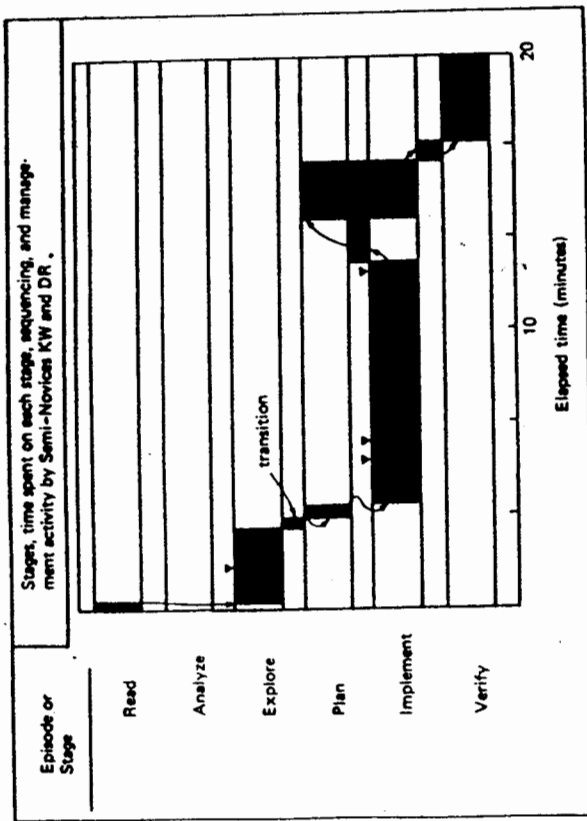


FIG. 8.4.

type. Figure 8.4 presents the graph of a relatively typical postinstruction attempt to solve a nonstandard problem. The entire solution, and a discussion of it, may be found in chapter 9 of Schoenfeld (1985). What follows is a brief outline of their attempt.

As Fig. 8.4 indicates, the students solved the problem they were asked to work. That's a nice bonus, but it's not central to the current discussion. What is important is the route the students took along the way to getting their answer. They began typically, reading the problem and then jumping into a solution attempt.⁶ But 2 minutes into the solution, one of the students commented that they might be barking up the wrong tree and that the attempt they were making was based on a possibly unfounded assumption. They stopped what they were doing and decided to try something else. Unfortunately, what they decided to try was also wrong. For 8½ minutes, they were involved in complicated calculations, which ultimately led nowhere. But about 12 minutes into the solution, they pulled up short once again. "No, we aren't getting anything here. We're just getting . . . [what we're doing is unjustified] . . . Let's start all over and forget about this." Within the next 2 minutes they found an approach

⁶As I mentioned earlier, old habits die hard.

that worked, solved the problem, and confirmed their solution's correctness.

Viewed from one perspective, the way this problem was solved was hardly "ideal." The students began with a leap into exploration, which could have been avoided. Soon after that they embarked on an 8½-minute-long wild goose chase that, like the first exploration, could have been forestalled if they had carefully examined what they planned to do. There is still a fairly sharp contrast between this graph and the graph of an expert's problem solution shown in Fig. 8.2. But to make such comparisons is unfair and misses the point. The real comparison is with the graph in Fig. 8.1.

As noted, the students did solve the given problem. Since solving nonstandard problems can be a function of luck and prior knowledge, the fact that they succeeded must to some degree be considered a happy accident; one cannot take credit for this aspect of their success. What is not accidental, however, is that the students had the *opportunity* to find their solution. If the problem session had been recorded before the course, there is a good chance that the students would have pursued their first guess for all of their allotted time. (Nearly 60% of their peers did, and they themselves did on problems recorded before the course.) And if they had escaped that trap, another and much more serious one was waiting for them: The detailed computations that occupied them for 8½ minutes. In my experience, students who become that deeply immersed in computations virtually never manage to extricate themselves from them. It is almost certain that, before the course, poor metacognitive skills would have guaranteed their failure. After the course, the possession of such skills could not guarantee success, but at least it could give them a fair shot at it. By avoiding one wild goose chase entirely and bringing the second to a halt, the students salvaged their solution attempt twice. The result was that they then had the opportunity to find an approach that solved the problem (and they did so). Viewed from that perspective, their solution was indeed expertlike. In this and in other ways, the instruction has been quite successful.

IN SUMMARY, A SEMITHEORETICAL COMMENTARY

The instructional techniques described in this chapter have been justified by common sense, a metaphor, and one powerful post hoc argument—they work. In this concluding discussion I would like to examine some of the reasons why they work. What follows is a speculative exploration of the role of social context in the development of metacognition. It is

intended more to establish an agenda of interesting items to think about than to say anything definitive.

One way to characterize efficient self-regulation is to say that the people who are good at it are the people who are good at arguing with themselves. In the transcript of problem solving session graphed in Fig. 8.2, for example, you can almost see the problem solver taking on different roles as he works the problem. At different times, he functions as an idea generator, a systematic planner, a critic, a "progress monitor," an advocate first for one particular point of view ("Maybe I should do it this way . . .") and then for another, and so on. To use a popular phrase from cognitive science, you see him function in his problem solving as a "society of mind"—putting forth multiple perspectives, weighing them against each other, and selecting among them. When this works well, it is highly productive. The idea generator suggests many interesting things for him to try as he works, while the critic and monitor can keep him from going off on wild goose chases.

Where do these skills come from? One point of view, pioneered by the Soviet psychologist L. S. Vygotsky in his *Thought and Language* (1962) and *Mind in Society* (1978), is that all higher order cognitive skills originate in, and develop by the internalization of, individuals' interactions with others.

Every function in the child's development appears twice: first, on the social level, and later on the individual level; first, *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals. (Vygotsky, 1978, p. 57)

Vygotsky hypothesizes that the potential for development at any time is limited to what he calls the "zone of proximal development (ZPD)," defined as follows. Working alone, the child may function up to a certain level. Working in collaboration with more capable peers, or perhaps with adult guidance, the child may function at a somewhat higher level. This middle ground, which the student is capable of reaching with some assistance but not on his own, is the ZPD. Vygotsky's thesis is that one acquires higher order skills by exercising those skills in the ZPD with the help of others and then internalizing those skills, that is, by mastering as an individual those skills for which one, at one time, needed support.

This perspective provides strong justification for the use of small groups in problem-solving contexts. Suppose the description of the good problem solver as one who argues (intelligently) with himself is right and that a large component of effective problem solving consists of advancing multiple perspectives, balancing them against each other, and proceeding

on the basis of what seems, on balance, to be the best option at the time. Almost by definition, small-group discussions (when they work well) result in the individual's working in his ZPD. While the individual might generate one possibility and go off in pursuit of it, a group might generate three or four—and precisely because they have more than one option, the group will have to decide among them. In consequence, the individual might have to formulate and defend one point of view, listen to and evaluate others, and finally take part in a group decision regarding which one(s) to pursue and for how long. These are precisely the self-regulation skills the individual needs to develop, and it is difficult to imagine a context in which they could develop more naturally.⁷

For some years this version of the Vygotskian hypothesis has been the primary theoretical support for my use of small groups. As noted earlier, the coaching metaphor provided additional justification. For the sake of completeness, here are two other arguments I (1983) have used:

Problem solving is not always a solitary endeavor. Students have little opportunity to engage in collaborative efforts, and this does not do them any harm.

Students are remarkably insecure, especially in a course of this nature. Working on problems in groups is reassuring: one sees that his fellow students are also having difficulty, and that they too have to struggle to make sense of the problems that have been thrown at them. (pp. 30–31)

I have come to think that such statements fail to do justice to the phenomena we have been discussing. The issue is much larger than these arguments would suggest, and it is primarily a cultural one.

Let us return to the topic of collaboration. The stereotype is of the mathematician alone in his office, scrawling on a pad long into the night. For this reason it is especially interesting to note how mathematicians themselves discuss the topic. Donald Albers' and Gerald Alexanderson's recent (1985) *Mathematical People: Profiles and Interviews* offers a collection of conversations with and about contemporary figures in mathematics. It is surprising how frequently the mathematicians mention collaboration, both in terms of influences on their work and in terms of the benefit they derive from working with others. In one case, "collaborative distance" has actually been formalized. Paul Erdős is a remarkably

⁷I have thought about slightly less natural small group interactions, designed to promote the growth of particular characters in the society of mind. Suppose you structured small groups so that individuals had (one or more) assigned roles—say as idea generator, planner, critic, monitor, etc. Making these role assignments overt might make the tasks more salient, and provide more direct practice at them. This seems like an interesting idea, but I haven't tried it. I suspect that making the right role assignments and getting the group dynamics to work might be decidedly non-trivial.

prolific mathematician who has produced some 900 papers, of which about 200 were coauthored. Each of his coauthors has an "Erdős number" of 1. Each of the people who has coauthored a paper with one of them has an Erdős number of 2, and so on. Einstein had an Erdős number of 2, and there is ongoing historical research to determine the Erdős number assigned to C. F. Gauss (1777-1855), one of the greatest mathematicians of all time.

Peter Hilton lays out the benefits of collaboration as follows (Albers & Alexanderson, 1985):

First I must say that I do enjoy it. I very much enjoy collaborating with friends. Second, I think it is an efficient thing to do because . . . if you are just working on your own [you may] run out of steam. . . . But with two of you, what tends to happen is that when one person begins to feel a flagging interest, the other one provides the stimulus. . . . The third thing is, if you choose people to collaborate with who somewhat complement rather than duplicate the contribution that you are able to make, probably a better product results. (p. 141)

The second and third reasons are rational and important, but Hilton lists the joy of collaboration first. Persi Diaconis says the following (Albers & Alexanderson, 1985):

There is a great advantage in working with a great co-author. There is excitement and fun, and it's something I notice happening more and more in mathematics. Mathematical people enjoy talking to each other. . . . Collaboration forces you to work beyond your normal level. Ron Graham has a nice way to put it. He says that when you've done a joint paper, both co-authors do 75% of the work, and that's about right. . . . Collaboration for me means enjoying talking and explaining, false starts, and the interaction of personalities. It's a great, great joy to me. (pp. 74-75)

The last quotation captures much of the essence of what it is to be a mathematician, or, better, to be a member of the mathematical community. Diaconis speaks of excitement, of mathematical people (note the very phrase!) taking pleasure in talking to each other; of talking and explaining, of the joy in false starts, and the interaction of personalities. Not only is it a "great, great joy"; I would argue that those interactions, and the sense of community—a culture of mathematics, if you will—are part of what sustains mathematics. And participation in that culture is how one comes to understand what mathematics is.

I remember discussing with some colleagues, early in our careers, what it was like to be a mathematician. Despite obvious individual differences, we had all developed what might be called the mathematician's point of view—a certain way of thinking about mathematics, of its value, of how it

is done. What we had picked up was much more than a set of skills; it was a way of viewing the world and our work. We came to realize that we had undergone a process of acculturation, in which we had become members of, and had accepted the values of, a particular community.⁸ As the result of a protracted apprenticeship into mathematics, we had become mathematicians in a deep sense (by dint of world view) as well as by definition (what we were trained in and did for a living).

Now, what does this have to do with ordinary, day-to-day mathematics instruction? I think a great deal. If the notion of culture as explored here makes sense, it explains why my problem-solving courses have been successful in the way that they have been—and why so many attempts at curriculum reform have failed.

My problem-solving courses have evolved in an interesting way. Early versions of the courses focused on the thinking tools I thought students needed for competent problem-solving performance. This meant giving training in particular heuristic strategies, teaching a prescriptive "managerial strategy" for self-regulation, and pointing directly to problems (or domains) where the "wrong" beliefs caused difficulty. Over time the managerial strategy disappeared (it was artificial) and was replaced by the more natural in-class techniques described above. Work on beliefs moved in the same direction. I began with "damage containment," identifying counterproductive beliefs and dealing with them on a case-by-case basis. Later I moved toward a natural environment, in which we worked as a group doing mathematics the "right" way.

With hindsight, I realize that what I succeeded in doing in the most recent versions of my problem-solving course was to create a microcosm of mathematical culture. Mathematics was the medium of exchange. We talked about mathematics, explained it to each other, shared the false starts, enjoyed the interaction of personalities. In short, we became mathematical people. It was fun, but it was also natural and felt right; it wasn't a separate "school experience" for a few hours a week. By virtue of this cultural immersion, the students experienced mathematics in a way that made sense, in a way similar to the way mathematicians live it. For that reason, the course has a much greater chance of having a lasting effect.

⁸Let me give just one example. To this day I have the vivid memory of Peggy Strait, who taught my undergraduate probability course at Queens College, going to the board to write the statement of a theorem. When she got to the statement of the main result she stopped and said "I never remember this result, but that's no problem; it's so easy to derive." She did just that, showing us why it made sense; then she finished writing the statement of the theorem. At that moment I saw how mathematics should be: if you really understand it, you don't have to memorize a lot, because you can figure it out. This became part of my sense of what mathematics is all about.

I argued earlier that the interactions among mathematicians, and the sense of community they support, are part of what sustains mathematics—that the practice of mathematics is a human endeavor and very much a cultural one. I also suggested that entry into that culture (or some culture that supports the same values) may be necessary to understand and appreciate mathematics. If this is right, it explains at least partially the failure of a whole slew of curriculum reform movements—moves for “relevance,” for the “new math,” for “basic skills,” and, I predict, for “problem solving.” Each of these curriculum reforms reflects an attempt to embed a selected aspect of mathematical thinking into what is an essentially alien culture, that of the traditional classroom. As long as the two cultures differ as radically as they do at present, it may be impossible for this kind of embedding to succeed. Fragments of the mathematical culture, in isolation, are likely to wither in the classroom for lack of support, or they will be so changed by their absorption into the classroom culture that they will not transfer back outside of it. This holds even for attempts to link schooling and the real world, such as the “relevance” movement and attempts to use “applied” problems in curricula. According to this view, those attempts didn’t have a chance.

Although these comments may sound pessimistic, my sense is just the opposite—that substantial optimism is warranted. If there is real substance to this notion of culture as the medium for knowledge transmission, then there is the potential for a research program that may lead us out of the “school learning doesn’t transfer to real life” dilemma. The idea, of course, would be to engender a culture of schooling that reflects the use of mathematical knowledge outside the school context. To achieve this we need to understand more about mathematical thinking and culture in at least two ways. In this chapter I have focused on elaborating aspects of mathematical thinking as seen from the mathematician’s point of view—what it is to live in a culture of mathematics. But the flip side of the coin, understanding the development of mathematical thinking in our culture at large, is equally crucial. There is a growing body of anthropological work on cognition in practice (see, e.g., Carraher, Carraher, & Schliemann, 1985; Lave, in press; Rogoff & Lave, 1984) indicating that people are very good at inventing the mathematics they need to carry out tasks that are really important to them. My sense is that the two cultures are compatible and that, when we understand enough about them, they can be reasonably conjoined. If this is the case, what we need is a program of “cultural design” for schooling. Understanding enough about the social contexts that promote the need to develop and understand mathematical ideas, and about the environments that support the growth and development of those ideas, may allow us to create classroom

cultures in which students *do* mathematics naturally. When that happens, the “transfer problem” will no longer be a problem.

REFERENCES

- Albers, D. J., & Alexander, G. L. (1985). *Mathematical people: Profiles and interviews*. Chicago: Contemporary Books.
- Brown, A. L. (1978). Knowing when, where, and how to remember: A problem of metacognition. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 77–165) (Vol. 1). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Brown, A. L., Bransford, J., Ferrara, R., & Campione, J. (1983). Learning, remembering, and understanding. In P. H. Mussen (Ed.), *Handbook of child psychology* (Vol. 3) (pp. 77–166). New York: Wiley.
- Carpenter, T. P., Lindquist, M. M., Matthews, W., & Silver, E. A. (1983). Results of the third NAEP mathematics assessment: Secondary school. *Mathematics Teacher*, 76(9), 652–659.
- Carraher, T. N., Carraher, D. W., & Schlemann, A. D. (1985). Mathematics in the streets and in the schools. *British Journal of Developmental Psychology*, 3, 21–29.
- Collins, A. L., & Brown, J. S. (in press). The new apprenticeship: Teaching students the craft of reading, writing, and mathematics. In L. B. Resnick (Ed.), *Cognition and instruction: Issues and agendas*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Lave, J. (in press). *Arithmetic practice and cognitive theory*. Cambridge: Cambridge University Press.
- Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. New York: Addison-Wesley.
- Pólya, G. (1981). *Mathematical discovery* (Combined paperback edition). New York: Wiley.
- Rogoff, B., & Lave, J. (1984). *Everyday cognition: Its development in social context*. Cambridge, MA: Harvard University Press.
- Schoenfeld, A. H. (1983). *Problem solving in the mathematics curriculum: A report, recommendations, and an annotated bibliography*. Washington, DC: Mathematical Association of America.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Schoenfeld, A. H. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Schoenfeld, A. H. (in press). When good teaching to bad results: The disasters of well taught mathematics courses. *Educational Psychologist*.
- Vygotsky, L. S. (1962). (E. Hanfmann & G. Vakar, Eds. & Trans.). *Thought and language*. Cambridge, MA: MIT Press and Wiley.
- Vygotsky, L. S. (1978). (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Trans.) *Mind in society*. Cambridge, MA: Harvard University Press.