What is chaos?

- Irregular motions arising in deterministic systems...
- …or, an exponential divergence of initially nearby trajectories in phase space.
- Timescale of divergence is the Lyapunov time.
- Examples: driven pendulum; double pendulum; gravitational 3-body problem.
A driven pendulum and its surface of section
Historical perspective

• “Is the Solar System stable?” is an old question.
• Stimulated the early development of topology.
• Detailed study of chaos only possible with the advent of computers.
• Now we can ask, “Is the Solar System chaotic?”
Conditions for chaos

- Stability if there is an integral of the motion for each degree of freedom.
- No such integral exists for the three-body problem.
- Non-analytic invariant curves exist for small perturbations with couplings far from resonance (KAM theorem) – not satisfied in the Solar System!
- Chaos arises where neighbouring resonances overlap – this underpins analytic work arising from perturbation theory.
Types of resonance

• Mean-motion resonance
  – Orbital periods in integral ratios.
  – For Jupiter, 4:3 and 5:4 mean-motion resonances overlap, as do all higher resonances.

• Secular resonance
  – Equality of apsidal or nodal rates.
  – Important within mean-motion resonances, e.g. Kirkwood gaps in asteroid belt.
Kirkwood gaps

- Low eccentricity objects increase eccentricity resulting in planet crossing.
- Even objects in stable high eccentricity orbits may be extracted by Mars.
- For the 3:1 resonance timescales are of order 10-100 Myr.
- Resonances more dense past 2:1 – “more chaos”.
Mysterious resonances

• Secondary resonances may contribute to odd behaviour at 2:1 and 3:2
  – Integral ratios of $P_{\text{apse}}$ and $P_{\text{lib}}$
  – Needs more realistic models.

• Gaps at fifth and sixth order resonances seem to be too pronounced
  – Ejection timescales of order the age of the Solar System.
  – Other factors may come into play
Beyond the asteroid belt

- Low surface density of asteroids between Jupiter and Saturn
  - Most objects ejected very quickly.
  - Very few survive past $\sim 10^7$ years.
- Similarly between the other outer planets, but lifetimes increase to $\sim 10^9$ years between Uranus and Neptune
Inner Solar System
Divergence in phase space
Kuiper Belt

• Disc of material beyond Neptune.
• Stable regions, with ejection times \(\sim 10^9\) years.
• Unstable regions, with ejection times \(\sim 10^7-10^9\) years.
• Mixture of regions and diffusion between them gives a reservoir of comets.
Dynamical Lifetimes of particles with initial l = 1°

Initial Eccentricity

q = 30 AU

q = 35 AU

2:3 3:5 4:7

24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60
Are planets safe?

• Big variations in eccentricity of Mercury – diffuses to e=1 in 2000 Gyr
• No dynamical barrier to ejection.
• Only Neptune and Pluto in two-body mean-motion resonance.
• Three-body resonances give ejection times $\sim 10^{18}$ yrs.
• Variations in obliquity of Mars.
Obliquity of Mars over 400 million years
Conclusions

• The Solar System is chaotic.
• \((\text{Crossing time}) \sim (\text{Lyapunov time})^{1.75}\) but with large spread.
• Two resonances overlap and cause an increase in eccentricity, and eventually orbit crossing
• Would expect stronger chaos in the early Solar System.
• Maverick objects have already been ejected – the Solar System is old, not stable!