

ASTR 1030: Homework 3

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Kepler's Laws (Kevin)

Kepler's Second Law states that when a planet orbits its parent star, its motion sweeps out equal areas in a given time,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \text{constant}, \quad (1)$$

which is a statement of the conservation of angular momentum. Kepler's Third Law states that the orbital period of the planet is related to its semi-major axis, a , by

$$T = \frac{2\pi}{\sqrt{G(M_s + M_p)}} a^{3/2}, \quad (2)$$

where M_s and M_p are the masses of the star and the planet, respectively, and G is Newton's gravitational constant. Here, we explore the fact that these Laws are not entirely independent — rather, the Third Law can be derived from the Second one.

1. **(40 points)** Recall that conic sections, as discussed in class, are intersections of a given plane with a cone. The eccentricity of any given conic section is

$$e = \sqrt{1 + \frac{2El^2}{\mu k^2}}, \quad (3)$$

which is stated without proof — i.e., you do not have to prove this equation to me to use it. The quantities E , l and μ are the energy, (magnitude of the) angular momentum and reduced mass of the system, respectively, while k is a constant we will determine later. First, we want to relate E and k with a . We do this by solving the quadratic equation,

$$r^2 + \frac{k}{E}r - \frac{l^2}{2\mu E} = 0. \quad (4)$$

Obtain two solutions (since it is a quadratic equation) to the equation above, r_1 and r_2 . (If you do not remember how to solve a quadratic equation, look it up.) Then obtain an expression for $a = (r_1 + r_2)/2$. Substitute your expression for a into equation (3) and show that

$$e = \sqrt{1 - \frac{l^2}{\mu ka}}. \quad (5)$$

2. **(10 points)** It is possible to show, using the conservation of angular momentum, that

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2}. \quad (6)$$

Again, we state this without proof. Substitute this equation into Kepler's Second Law and obtain an expression for the area, A , using

$$A = \int_0^T \frac{dA}{dt} dt. \quad (7)$$

(Hint: l and μ are constants of the system.)

3. **(30 points)** The answer for Q2 relates A and T . Using the mathematical properties of the ellipse, which we discussed in class, along with equation (5), show that

$$T = 2\pi \sqrt{\frac{\mu}{k}} a^{3/2}. \quad (8)$$

(Hint: You get 20 points just for stating two *relevant* equations describing the properties of an ellipse which pertains to this problem.)

4. **(10 points)** Newton's Law for the force between M_s and M_p states that $F = -k/r^2$, where r is the distance between the star and the planet. What is k ? (Hint: You do not need to prove this. I just want you to look it up.)
5. **(10 points)** The reduced mass of the system is given by

$$\mu = \frac{M_s M_p}{M_s + M_p}. \quad (9)$$

Substitute this expression and your answer for k into equation (8). What do you get for T ? Is this Newton's or Kepler's version of Kepler's Third Law?