In this exercise, you will take a $T(m)$ model (e.g., the FALC, which you already have) and “decompose” it into the relevant density stratification: $n_e, n_p, n_H$. Why do we do this? It’s because the densities are the fundamental quantities needed to calculate collisional excitation and ionization rates, opacities, and pressure broadening coefficients. Model decomposition is the first step in solving a line transport problem. If one is trying to match a spectral line profile, say the Ca II H and K features, one often will empirically adjust a test $T(m)$ stratification until the simulated spectral profile matches the observed tracings. Each time one alters the temperature stratification, however, one must recompute the density/ionization structure of the model. This is relatively trivial in LTE, but LTE is a lousy approximation for the solar chromosphere, where overionization of hydrogen can occur owing to the presence of significant solar continuum emission shortward of the Balmer edge, and the maintenance of a significant $n = 2$ population owing to a heavily saturated Ly$\alpha$ resonance line. In fact, the proper treatment of the hydrogen ionization problem in a stellar chromosphere is one of the toughest RT problems there is, and yet one must solve it every time one simply constructs a chromospheric model. Only an unusually cruel instructor would set the full problem to a class at this stage of their training! There is, however, a simple shortcut that can be used to get around most of the difficulties of the chromospheric hydrogen ionization problem; which hopefully produces a decent approximation to the ionization structure, without having to tackle the nasty transport problem in a multilevel hydrogen atom. The shortcut is to assume that the Ly$\alpha$ transition throughout most of the chromosphere is so opaque that the $n = 1$ and $n = 2$ levels are driven into “detailed balance,” namely that the ratio of the populations is given by the LTE value

$$n_1/n_2 \equiv (n_1/n_2)^*.$$  \hspace{1cm} (1)

Note that this applies to the ratio of the populations, not that the actual populations are the LTE values. What detailed balance means in this context is that the Ly$\alpha$ excitation and deexcitation are strongly coupling the two levels together, and the fact that the Ly$\alpha$ line is so opaque means that the radiative excitation $\hat{J}$ will be well described by isotropic local $B_\nu(T)$, which we will remember implies that the population ratio will be Boltzmann (e.g., eq. 1).

The other simplification is to assume that the dominant rates out of and into the Balmer level $n = 2$ are photoionizations and radiative recombinations; i.e., that the collisional ionization and 3-body collisional recombination rates can be ignored in comparison. This actually isn’t too bad an approximation for the low-density chromospheric layers; but, wouldn’t apply for the deep photosphere where the densities are several orders of magnitude higher. (Thankfully, we aren’t too worried about the latter situation because the high collision rates will enforce LTE ionization balance, and we already know—or should know—how to solve the LTE problem.) With the radiation-dominated assumption, the statistical equilibrium relation for $n = 2$, in the “1-level plus continuum” approximation (ignoring for the moment $n = 1$ which we know is slaved to $n = 2$ by detailed balance) is

$$n_2 R_{2\kappa} = n_\kappa R_{\kappa 2}. $$  \hspace{1cm} (2)

Here, $n_\kappa$ is the population of the hydrogen continuum state (don’t confuse $\kappa$ with the opacity expressed in cm$^2$ gram$^{-1}$). What is the hydrogen continuum state, you ask? I hope you all answered “protons,” because that’s what it is. So, you can replace the “$\kappa$” with a “p” if that is more transparent to you. The “$Rs$” in eq. 2 are the radiative rates. The photoionization rate (on lhs eq. 2) can be
written (for hydrogen)

\[ R_{2\kappa} = \alpha_0 \int_{\nu_0}^{\infty} \nu^{-3} \left( J_\nu / h\nu \right) d\nu, \]  

(3)

where \( J_\nu \) is the mean intensity of the radiation field in the Balmer continuum, and the coefficient \( \alpha_0 \) out in front contains all of those pesky cross sections and physical constants. It really helps to simplify eq. 3 by replacing \( J_\nu \) as follows:

\[ J_\nu \sim \omega B_\nu(T_{\text{rad}}). \]  

(4)

This is a very common type of substitution. We do it because the hydrogenic (cross section with \( \nu^{-3} \) frequency dependence) integral in eq. 3 can be solved analytically, in terms of an exponential integral (remember those?), for a Planckian radiation distribution. In eq. 4, the \( \omega \) is a “dilution function” which represents the fact that the assumed Planckian radiation field at the surface \( B_\nu(T_{\text{rad}}) \), parameterized by a “radiation temperature” \( T_{\text{rad}} \), can be isotropic (\( \omega \equiv 1 \)) in the high density, opaque photospheric layers, but become highly anisotropic out in the chromosphere, where a flood of radiation is coming up from below, but essentially nada is coming in from the outside. We’ve seen this before many times, and you already should know what \( \omega \) will be in this situation. If you guessed \( \frac{2}{3} \), you would be wrong! It’s \( \omega = \frac{1}{2} \), of course. So, in practice, we would have to decide what an appropriate value of \( T_{\text{rad}} \) is, and impose some reasonable depth-dependence on \( \omega \) to ensure that the two limiting values (\( \frac{1}{2} \) in the chromosphere, and 1 in the photosphere) are attained. Actually, we have to be a bit more careful, because when \( \omega \) approaches 1 that means the radiation field is becoming isotropic and the medium is becoming opaque, and in that case \( T_{\text{rad}} \to T \). A reasonable value of \( T_{\text{rad}} \) for the Sun is 5100 K. It was determined by integrating the observed solar UV spectrum down from the Balmer edge (\( \sim 3650 \) Å) against the \( \nu^{-3} \) hydrogenic cross section to get \( R_{2\kappa}(\text{obs}) \), and then inverting eq. 3 with \( \frac{1}{2} B_\nu(T_{\text{rad}}) \) replacing \( J_\nu \) to solve for an empirical value of \( T_{\text{rad}} \). For the solar chromosphere model we are going to decompose, the best strategy is to take \( T_{\text{rad}} = 5100 \) K and \( \omega = \frac{1}{2} \) throughout the chromosphere, down past the temperature minimum \( T_{\text{min}} \), and into the photosphere proper down to the point where the local temperature begins to exceed \( T_{\text{rad}} \). Deeper than that point, set \( T_{\text{rad}} = T \) and \( \omega = 1 \). The idea is that in the low opacity chromospheric layers, the Balmer radiation field is whatever is coming up from the photosphere beneath, but diluted by the fact that there is no incoming radiation from the outside. Down in the photosphere itself, however, the Balmer radiation field eventually saturates to the local Planck function under the joint influences of isotropy and high opacity. This is a pretty crude approximation: we’ll see how well—or poorly—it works.

That takes care of the photoionization part of the problem. What about the radiative recombination rate? It promises to be a major headache, eh? Ah, but we have an important ally here: detailed balance, again! Thanks to our familiarity with TE enclosures, we know that the radiative recombination rate must be related to the LTE photoionization rate as follows:

\[ n_2^* R_{2\kappa}^* = n_\kappa^* R_{\kappa 2}^*. \]  

(5)

However, the radiative recombination rate is a collisional process (involving a \( p \) and \( e^- \)) and thus \( R_{\kappa 2} \equiv R_{\kappa 2}^* \). So, we can write

\[ R_{\kappa 2} = (n_2/n_\kappa)^* R_{2\kappa}^*. \]  

(6)

Now, we will recognize from just shortly ago, that \( R_{2\kappa}^* \) must be eq. 3 with \( J_\nu \) replaced by \( B_\nu(T) \). This is very helpful, because we see that we can rewrite eq. 2 as

\[ n_2/n_\kappa = (n_2/n_\kappa)^* \frac{R_{2\kappa}^*(T)}{R_{2\kappa}(T_{\text{rad}}, \omega)}. \]  

(7)
and when all is said and done, the nasty $\alpha_0$ parameter, which is tough to calculate trying to maintain consistent units, will cancel between numerator and denominator. Yes, when in doubt, take a ratio! What’s left are the ionization integrals. These can be evaluated as $\omega \text{EXPINT}(39500/T_{\text{rad}})$, where the exponential integral function will be provided as separate IDL routine.

So, let’s take a quick inventory. We have the ratio of $n_1/n_2$ from eq. 1. We also have the ratio $n_2/n_\kappa$ from eq. 5 (noting that the LTE ratio $(n_2/n_\kappa)^*$ is just a Saha factor [but contains the electron density $n_e$]: you should be able to rewrite that factor in terms of $(n_1/n_\kappa)^*$ pretty trivially, eh?). We know, further, that $n_1 + n_2 + n_\kappa = n_H$. That takes care of the hydrogen density for the moment.

Just for fun, let’s assume that we have a mixture mostly of hydrogen, but with 10% helium (abundance relative to H by number, not weight), and 0.010% of a trace metal, call it Robynium (Ry for short). Ry has an atomic weight of 36 AMU, and a first ionization potential of 7.9 eV. The neutral atom has a ground state with statistical weight $g_1 = 10$, and excited states at $\Delta E_2 = 3$ eV with $g_2 = 20$, and $\Delta E_3 = 5$ eV with $g_3 = 40$; the ion has a ground state with statistical weight $g_1^+ = 20$, and excited states at $\Delta E_2^+ = 5$ eV with $g_2^+ = 10$, and $\Delta E_3^+ = 7$ eV with $g_3^+ = 10$.

[Calculation Numero Uno]: Write a couple of lines of code to calculate the partition functions $U_I$ and $U_{II}$ for Ry.

OK, we’re ready to go. First, we have to get the total density $n_{\text{tot}}$ from the column mass density $m$:

$$n_{\text{tot}} = \frac{gm}{kT}.$$  \hspace{1cm} (8)

This comes, of course, from the condition of hydrostatic equilibrium ($P_{\text{gas}} = gm$) and the perfect gas law equation of state. Now, you’ll have to write down one equation for $n_{\text{tot}}$ in terms of the constituent particles, and a second equation for the electron density $n_e$ in terms of the electron donors (don’t forget the metal Ry: you can assume that it can exist as either Ry or Ry$^+$, and that the Ry ionization equilibrium is in LTE throughout the atmosphere; you can ignore helium ionization). The charge balance equation for $n_e$ is nonlinear because the various Saha factors contain $n_e$, so you probably will have to develop some kind of iterative scheme to solve the particle conservation and charge balance equations simultaneously; getting in the end values for the key densities $n_H$, $n_p$, and $n_e$ at each $[T,m]$ point in the atmosphere, consistent with the Balmer photoionization approximation laid out earlier. (Hint: a good choice for a solution scheme is to linearize the equations in $n_e$ and use a Newton-Raphson type of iteration to home in on the self-consistent value of $n_e$. If you take this approach, it is wise to consider two regimes for starting values of $n_e$: $T > 6000$ K, where the electrons are mostly from hydrogen and you can take $n_e = n_p$ in LTE as an initial guess; and $T < 6000$ K, where the ionization is dominated by Ry, and one simply can take $n_e \sim A_{Ry} n_H$, where $A_{Ry}$ is the abundance of Ry given previously.)

[Calculation Numero Dos]: Write a bunch more lines of code to calculate the density stratification at each depth in the FALC model, with the above approximations. (Harder than it sounds, and it sounds hard already!)

[Question Numero Tres]: Compare your density stratification with that provided in the FALC tabulation. What do you conclude? That’s it: have fun!